CHAPTER

3

INTEGRALS

Syllabus coverage

TOPIC 3.2: INTEGRALS

Anti-differentiation

3.2.1 identify anti-differentiation as the reverse of differentiation

3.2.2 use the notation $\int f(x)dx$ for anti-derivatives or indefinite integrals

3.2.3 establish and use the formula $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$ for $n \neq -1$

3.2.6 identify and use linearity of anti-differentiation

3.2.7 determine indefinite integrals of the form $\int f(ax - b)dx$

3.2.8 identify families of curves with the same derivative function

3.2.9 determine $f(x)$, given $f'(x)$ and an initial condition $f(a) = b$

Definite integrals

3.2.10 examine the area problem and use sums of the form $\Sigma_i f(x_i)\,\delta x_i$ to estimate the area under the curve $y = f(x)$

3.2.11 identify the definite integral $\int_a^b f(x)dx$ as a limit of sums of the form $\Sigma_i f(x_i)\,\delta x_i$

3.2.12 interpret the definite integral $\int_a^b f(x)dx$ as area under the curve $y = f(x)$ if $f(x) > 0$

3.2.13 interpret $\int_a^b f(x)dx$ as a sum of signed areas

3.2.14 apply the additivity and linearity of definite integrals

Fundamental theorem

3.2.15 examine the concept of the signed area function $F(x) = \int_a^x f(t)\,dt$

3.2.16 apply the theorem: $F'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$, and illustrate its proof geometrically

3.2.17 develop the formula $\int_a^b f'(x)dx = f(b) - f(a)$ and use it to calculate definite integrals

Applications of integration

3.2.18 calculate total change by integrating instantaneous or marginal rate of change

3.2.19 calculate the area under a curve

3.2.20 calculate the area between curves determined by functions of the form $y = f(x)$

3.2.21 determine displacement given velocity in linear motion problems

3.2.22 determine positions given linear acceleration and initial values of position and velocity

3.1 The anti-derivative

3.1

The **anti-derivative** or **integral** or **primitive** of a function $f(x)$ is the function $F(x)$ whose **derivative** is $f(x)$. In other words, if $F(x)$ is the anti-derivative of $f(x)$, then $F'(x) = f(x)$. Finding the anti-derivative is called **anti-differentiation** or **integration** and reverses the process of finding the derivative. When we differentiate a **constant** term (number), we get 0, so when anti-differentiating, we must include '$+ c$' in the answer, where c stands for any number, called the **constant of integration**.

The general anti-derivative of $2x$ is $x^2 + c$, where c is a constant.

Algebraically, this is written as $\int 2x\,dx = x^2 + c$.

This is called the anti-derivative of $2x$, or the integral or primitive of $2x$.

- The symbol $\int$ is read as 'the integral of'.
- 'dx' means 'with respect to x'.

The integral of ax^n

$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c \text{ for } n \neq -1$$

where ax^n is called the **integrand**.

For any power function, we say 'add one to the power, divide by the new power'.

Video playlist The anti-derivative

Worksheet Anti-derivatives 1

WORKED EXAMPLE 1 Finding the anti-derivative

Find the anti-derivative of the expression $3x^2 + 2x - 4$.

Steps	Working
1 Write the function as a derivative.	$\frac{dy}{dx} = 3x^2 + 2x - 4$
2 Integrate each term using the formula $\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$.	$= \int (3x^2 + 2x - 4)\,dx$ $= \frac{3x^3}{3} + \frac{2x^2}{2} - 4x + c$ **Exam hack** You can check your answer by differentiating it to see whether you get the original expression.
3 Simplify the expression.	$= x^3 + x^2 - 4x + c$

The integral of $(ax + b)^n$

Worksheet
The chain rule

Consider the derivative of $y = (2x - 1)^3$.

Let $y = u^3$

$$\frac{dy}{du} = 3u^2$$

Also $u = 2x - 1$

$$\frac{du}{dx} = 2$$

Using the chain rule

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= 3u^2 \times 2\\ &= 6u^2\\ &= 6(2x-1)^2\end{aligned}$$

So in reverse, $\int 6(2x - 1)^2 dx = (2x - 1)^3 + c$

so $\int (2x - 1)^2 dx = \frac{1}{6}(2x - 1)^3 + c$.

This matches the general statement, 'add one to the power, divide by the new power', but here we also divide by the derivative of the 'inner' expression: $2x - 1$.

When integrating $(2x - 1)^2$, we divide by 3 (the power + 1) and by 2 (the coefficient of x).

Now consider the derivative of any power of $(ax + b)$, such as $y = (ax + b)^{n+1}$.

Let $y = u^{n+1}$ where $u = ax + b$ and use the chain rule.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= (n+1)u^n \times a\\ &= a(n+1)u^n\\ &= a(n+1)(ax+b)^n\end{aligned}$$

So in reverse, $\int a(n + 1)(ax + b)^n dx = (ax + b)^{n+1} + c$,

so $\int (ax + b)^n dx = \frac{1}{a(n+1)}(ax + b)^{n+1} + c$.

When integrating $(ax + b)^n$, we divide by $n + 1$ (the power + 1) and by a (the coefficient of x).

The integral of $(ax + b)^n$

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

Remember this as a **reverse chain rule**.

Exam hack

Many students forget to use this rule in examples such as $\int \frac{1}{(5x-3)^2} dx$. This becomes very usefully, $\int (5x - 3)^{-2} dx$, now of the form $\int (ax + b)^n dx$.

WORKED EXAMPLE 2 Finding the integral of $(ax + b)^n$

Find $\int (3x + 2)^4 \, dx$.

Steps	Working
1 Integrate using $\int (ax+b)^n \, dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + c$.	$\int (3x+2)^4 \, dx = \dfrac{1}{3(4+1)}(3x+2)^{4+1} + c$
2 Simplify.	$= \dfrac{1}{15}(3x+2)^5 + c$

We can skip Step 1 and go straight to the answer.

WORKED EXAMPLE 3 Finding y given $\dfrac{dy}{dx}$ and a point

Find y if $\dfrac{dy}{dx} = x^2 - x + 4$ and $x = 1$ when $y = 0$.

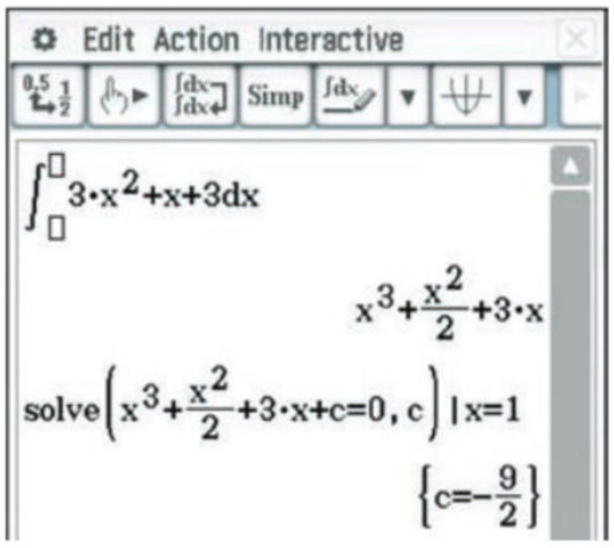
For this question, we can use given information to find the value of c.

Steps	Working
1 Write the expression as a derivative.	$\dfrac{dy}{dx} = x^2 - x + 4$
2 Integrate each term and simplify.	$y = \dfrac{x^3}{3} - \dfrac{x^2}{2} + 4x + c$
3 Find the value of c by substituting $x = 1$, $y = 0$.	When $x = 1$, $y = 0$: $0 = \dfrac{1^3}{3} - \dfrac{1^2}{2} + 4(1) + c$ $0 = \dfrac{23}{6} + c$ $c = -\dfrac{23}{6}$
4 State the answer including the value of c.	$y = \dfrac{x^3}{3} - \dfrac{x^2}{2} + 4x - \dfrac{23}{6}$

USING CAS 1 Finding y given $\frac{dy}{dx}$ and a point

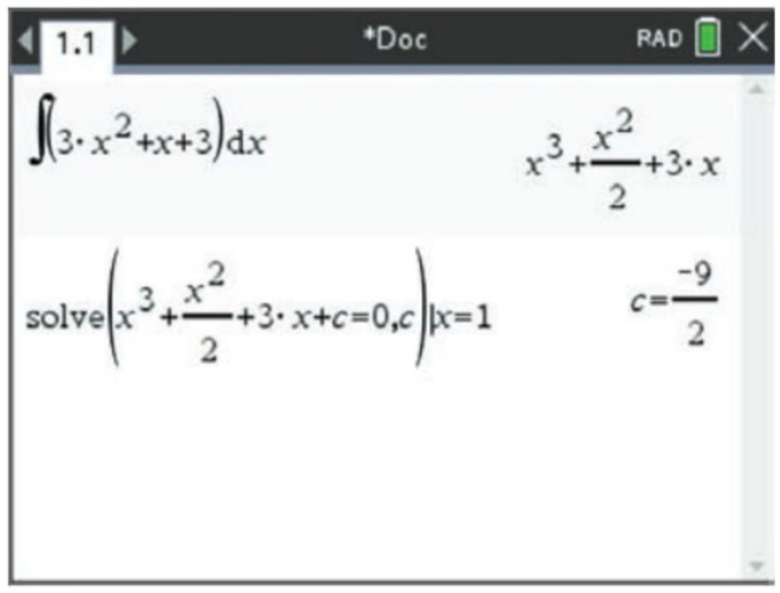
Find y if $\frac{dy}{dx} = 3x^2 + x + 3$ and $x = 1$ when $y = 0$.

ClassPad



1. Highlight the expression then tap **Interactive > Calculation >** $\int$.
2. Add **+c** to the expression as it is not included in the solution.
3. Set the expression including the **+c** equal to **0** and solve by including the condition that **x = 1**.
4. The value of **c** will be displayed.

TI-Nspire



1. Press **menu > Calculus > Integral**.
2. Enter the derivative expression followed by **dx**.
3. Add **+c** to the expression as it is not included in the solution.
4. Set the expression including the **+c** equal to **0** and solve by including the condition that **x = 1**.
5. The value of **c** will be displayed.

The solution is $y = x^3 + \frac{x^2}{2} + 3x - \frac{9}{2}$.

EXERCISE 3.1 The anti-derivative

ANSWERS p. 392

Mastery

1 WORKED EXAMPLE 1 Find the anti-derivative of the expression $x^3 + 3x^2 - 4x$.

2 WORKED EXAMPLE 2 Find $\int (4x - 1)^3 dx$.

3 Determine the anti-derivative for each of the expressions.

a $x^2 - 3x + 2$ **b** $(x - 3)(2x + 4)$ **c** $\frac{x^2 - 2x}{x}$ **d** $\sqrt{x} - \frac{1}{x^2} - 3$

e $\sqrt{(2x - 3)}$ **f** $\sqrt{x}(x^2 - 2x + 3)$ **g** $\frac{1}{(2x - 3)^2}$ **h** $\sqrt[3]{3x - 4}$

Exam hack

There is no reverse of the product or quotient rule. Try simplifying the expression prior to anti-differentiating.

4 WORKED EXAMPLE 3 Determine the function $f(x)$ if $f'(x) = 2x$ and $y = 1$ when $x = 2$.

5 Using CAS 1 Determine the function y if $\dfrac{dy}{dx} = 2x + 4$ and $y = 1$ when $x = 2$.

6 Determine the function $f(x)$ if $f'(x) = \dfrac{4}{\sqrt[3]{4-2x}}$ and $f(-2) = 10$.

3.1

Calculator-free

7 (2 marks) Determine the anti-derivative of the expression $3x^2 + 4x^3 - 2$.

8 (2 marks) Determine $\displaystyle\int \frac{1}{(3x+4)^4}\,dx$.

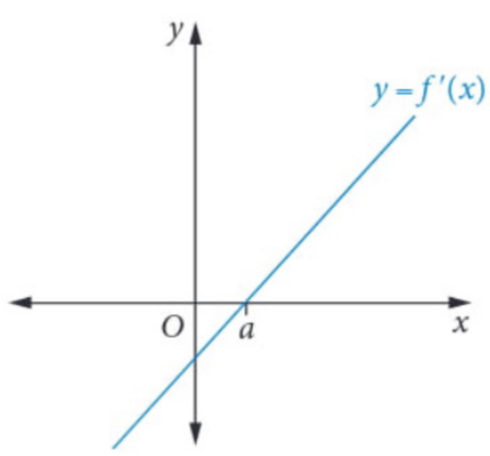
9 (2 marks) Find the anti-derivative of $(4-2x)^{-5}$ with respect to x.

10 (2 marks) Let $f'(x) = 3x^2 - 2x$ such that $f(4) = 0$. Determine $f(x)$.

11 (2 marks) If a and b are positive integers and $f'(x) = ax^2 - bx$, draw a possible sketch of $f(x)$.

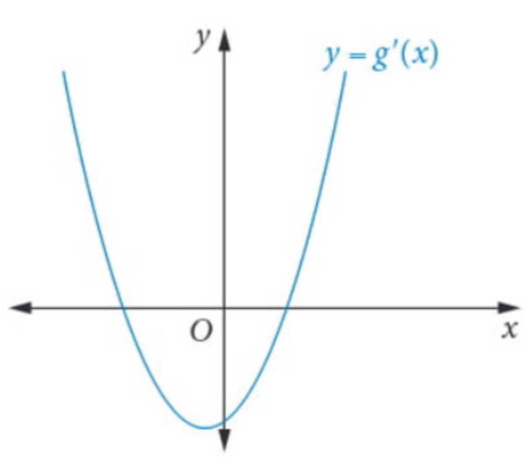
12 (2 marks) For each graph of a function, sketch a possible graph of its anti-derivative.

a



(1 mark)

b



(1 mark)

13 (2 marks) Determine $\displaystyle\int \frac{2x-3}{\sqrt{x^2-3x}}\,dx$.

Calculator-assumed

14 (2 marks) Find $f(x)$ given that $f(1) = -\dfrac{7}{4}$ and $f'(x) = 2x^2 - \dfrac{1}{4}x^{-\frac{2}{3}}$.

15 (2 marks) Find an anti-derivative of $\dfrac{1}{(2x-1)^3}$ with respect to x.

16 (2 marks) Let $f'(x) = \dfrac{2}{\sqrt{2x-3}}$. If $f(6) = 4$, determine $f(x)$.

17 (2 marks) If $f'(x) = g'(x) + 3$, $f(0) = 2$ and $g(0) = 1$, show that $f(x) = g(x) + 3x + 1$.

3.2 Approximating areas under curves

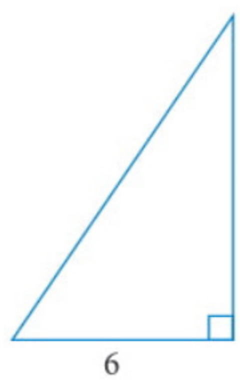
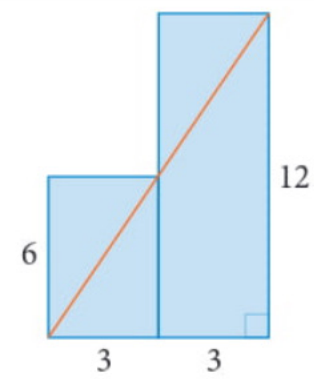
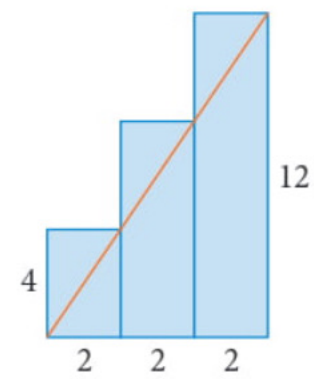
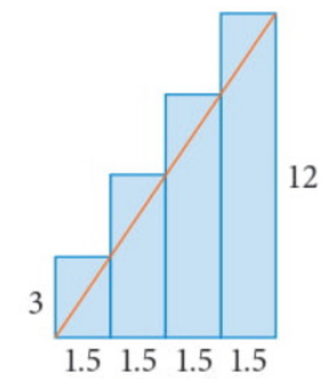
Video playlist Approximating areas under curves

Worksheet Areas using rectangles

The area under the graph of a function can provide important information in fields such as surveying, physics and the social sciences.

For example, the area under a speed graph shows the distance travelled.

We can estimate the area under a graph using a series of rectangles. The more rectangles we use, the more accurate the area will be. For example, the triangle below has an exact area of 36 units^2. To the right of this, the triangle has been approximated by 2, 3 and 4 rectangles.

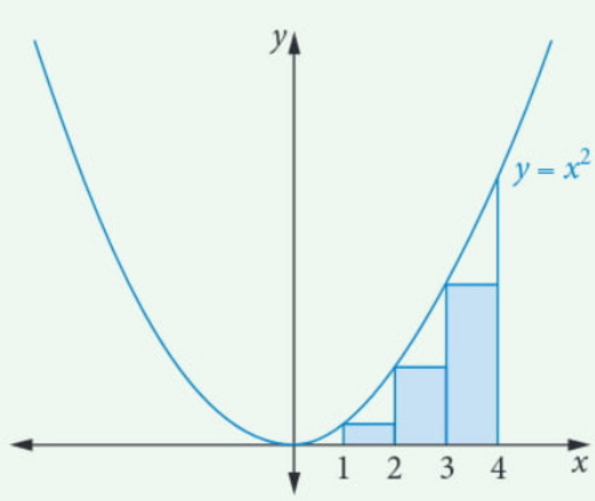
Actual area	Approximate area		
	 Using 2 rectangles	 Using 3 rectangles	 Using 4 rectangles
$\frac{1}{2} \times 6 \times 12 = 36$	$3 \times 6 + 3 \times 12 = 54$	$2 \times 4 + 2 \times 8 + 2 \times 12 = 48$	$1.5 \times 3 + 1.5 \times 6 + 1.5 \times 9 + 1.5 \times 12 = 45$

Notice that as the number of rectangles increases, the approximate area gets closer to the actual area of the triangle, in this case, 36 units^2.

We can use a similar idea to approximate the area under curves. When finding the area under any curve, we can draw a series of rectangles or 'vertical slices' to approximate the area.

WORKED EXAMPLE 4 Underestimating the approximate area under the curve

Find an approximation to the area under the curve $y = x^2$ between $x = 1$ and $x = 4$ using the sum of 3 rectangles shown below.



These rectangles are called lower rectangles because their tops touch the curve on their left corner. This is an underestimate of the area, because of the gaps between the tops of the rectangles and the curve.

Steps	**Working**
1 Find the height of each rectangle.	$f(1) = 1^2 = 1$ $f(2) = 2^2 = 4$ $f(3) = 3^2 = 9$
2 Find the area of each rectangle.	$A_1 = 1 \times 1 = 1$ $A_2 = 1 \times 4 = 4$ $A_3 = 1 \times 9 = 9$
3 Add the areas.	$A = 1 + 4 + 9 = 14$ The area is approximately 14 units^2.

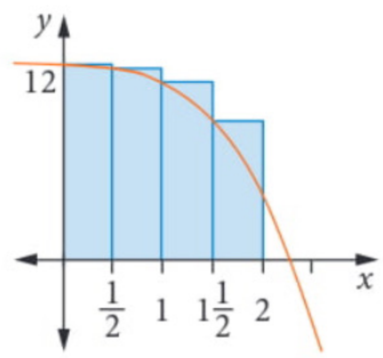
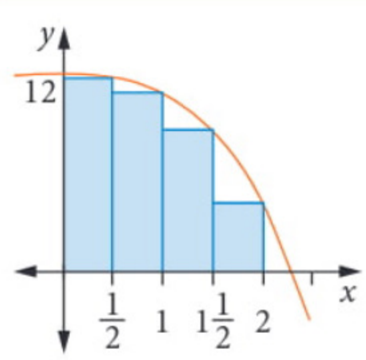
To improve the accuracy of the estimate, an overestimation using rectangles can also be done, and the two values averaged.

The accuracy can be further improved by using more rectangles of a smaller width.

WORKED EXAMPLE 5 Approximation using rectangles

Find an approximation to the area under the curve $y = 12 - x^3$ between $x = 0$ and $x = 2$ with the width of the rectangles being $\frac{1}{2}$ unit, using

a an overestimation of the area **b** an underestimation of the area.

Steps	Working
a 1 Sketch the graph. Draw the rectangles so that each touches the curve at the top left.	
2 State the x value for each rectangle.	The values are at 0, $\frac{1}{2}$, 1 and $\frac{3}{2}$.
3 Find the height of each rectangle.	$f(0) = 12$ $f\left(\frac{1}{2}\right) = \frac{95}{8}$ $f(1) = 11$ $f\left(\frac{3}{2}\right) = \frac{69}{8}$
4 Find the area of each rectangle.	$A_1 = \frac{1}{2} \times 12 = 6$ $A_2 = \frac{1}{2} \times \frac{95}{8} = \frac{95}{16}$ $A_3 = \frac{1}{2} \times 11 = \frac{11}{2}$ $A_4 = \frac{1}{2} \times \frac{69}{8} = \frac{69}{16}$
5 Add the areas.	$A = 6 + \frac{95}{16} + \frac{11}{2} + \frac{69}{16} = \frac{87}{4}$ Total area of left rectangles = $\frac{87}{4}$ units^2
b 1 Sketch the graph. Draw the rectangles so that each touches the curve at the top right.	
2 State the x value for each rectangle.	The values are at $\frac{1}{2}$, 1, $\frac{3}{2}$ and 2.

3 Find the height of each rectangle.

$f\left(\frac{1}{2}\right) = \frac{95}{8}$

$f(1) = 11$

$f\left(\frac{3}{2}\right) = \frac{69}{8}$

$f(2) = 4$

4 Find the area of each rectangle.

$A_1 = \frac{1}{2} \times \frac{95}{8} = \frac{95}{16}$

$A_2 = \frac{1}{2} \times 11 = \frac{11}{2}$

$A_3 = \frac{1}{2} \times \frac{69}{8} = \frac{69}{16}$

$A_4 = \frac{1}{2} \times 4 = 2$

5 Add the areas.

$A = \frac{95}{16} + \frac{11}{2} + \frac{69}{16} + 2 = \frac{71}{4}$

Total area of right rectangles = $\frac{71}{4}$ units^2

The actual area under the curve must be between $\frac{71}{4}$ and $\frac{87}{4}$ units^2. A more accurate approximation would be to take the mean of the underestimation and overestimation. In the above example this would be

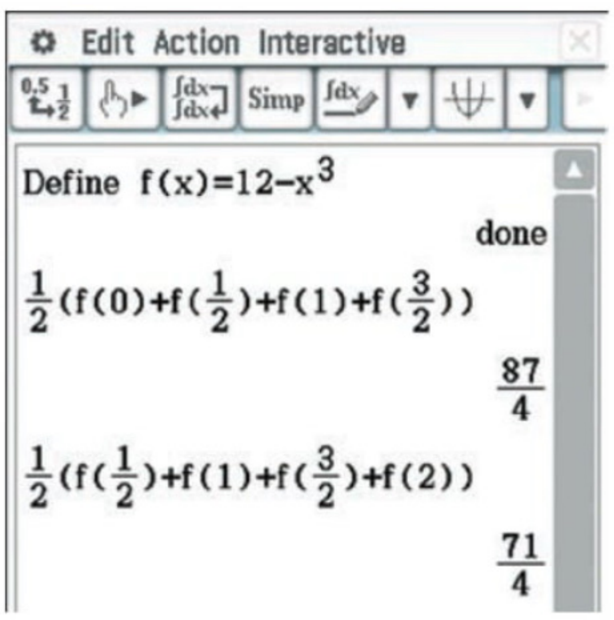
$$\left(\frac{87}{4} + \frac{71}{4}\right) \div 2 = \frac{79}{4} = 19.75 \text{ units}^2$$

We can calculate the above areas efficiently using CAS.

USING CAS 2 Approximating areas under curves

Find an approximation to the area under the curve $y = 12 - x^3$ between $x = 0$ and $x = 2$ with the width of the rectangles being $\frac{1}{2}$ a unit, using an overestimation and underestimation of the area.

ClassPad



1 Define **f(x)** as shown above.

2 Calculate the sum of the rectangles using the overestimate

3 Calculate the sum of the rectangles using the underestimate.

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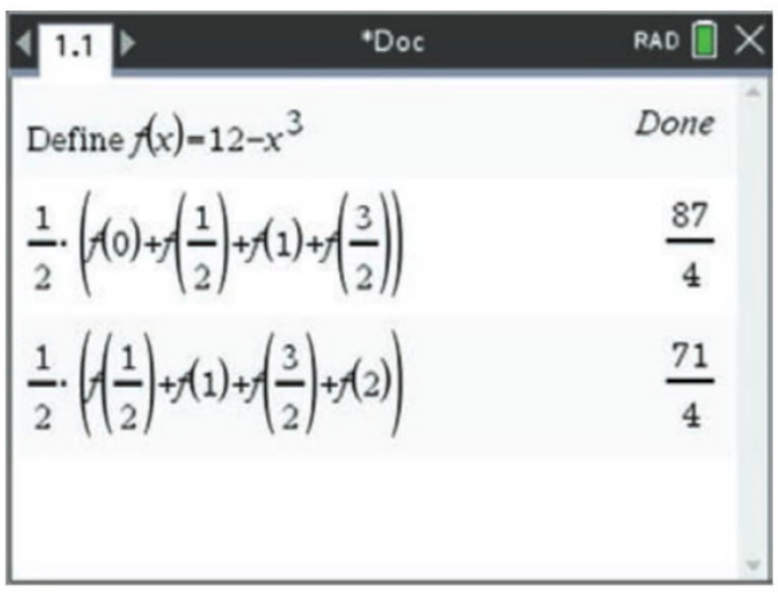

1 Define **f(x)** as shown above.

2 Calculate the sum of the 4 rectangles using the overestimate.

3 Calculate the sum of the 4 rectangles using the underestimate.

The overestimated area is $\frac{87}{4}$ units^2 and the underestimated area is $\frac{71}{4}$ units^2.

EXERCISE 3.2 Approximating areas under curves

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Recap

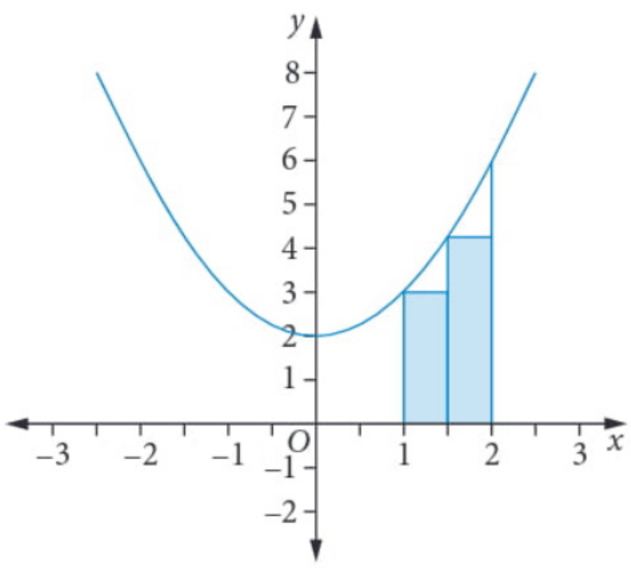
1 State the anti-derivative of $x^2 + 3x$.

2 Find $f(x)$ if $f'(x) = 2x - x^{\frac{2}{3}}$ and $f(1) = -1$.

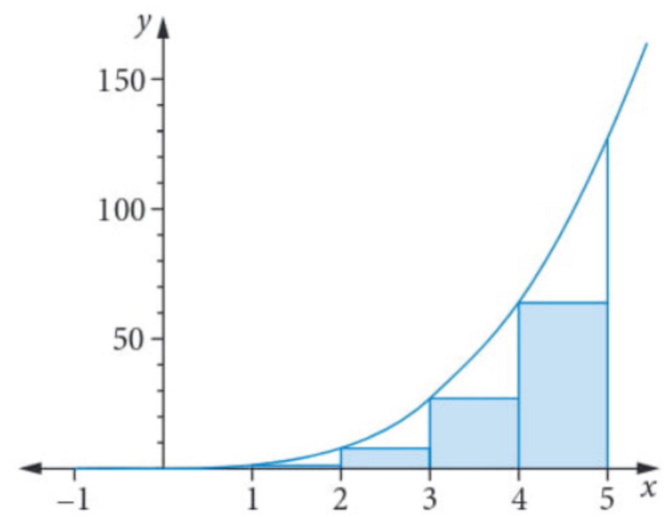
Mastery

3 WORKED EXAMPLE 4 Use the rectangles shown to find an approximation to the area under the curve. Give your answer correct to two decimal places.

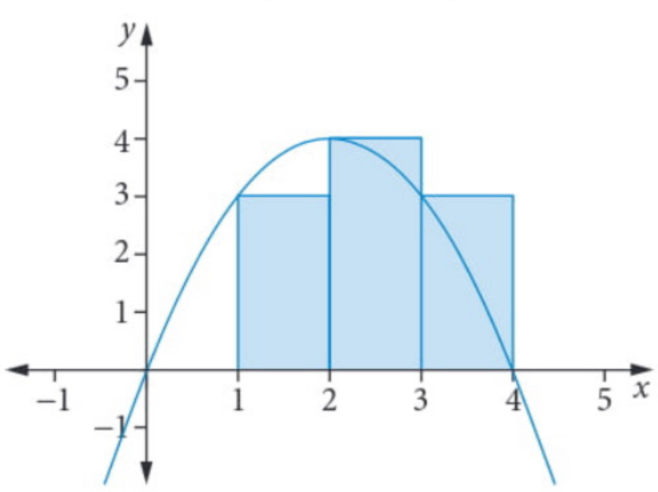
a $y = x^2 + 2$ between $x = 1$ and $x = 2$, using 2 rectangles.



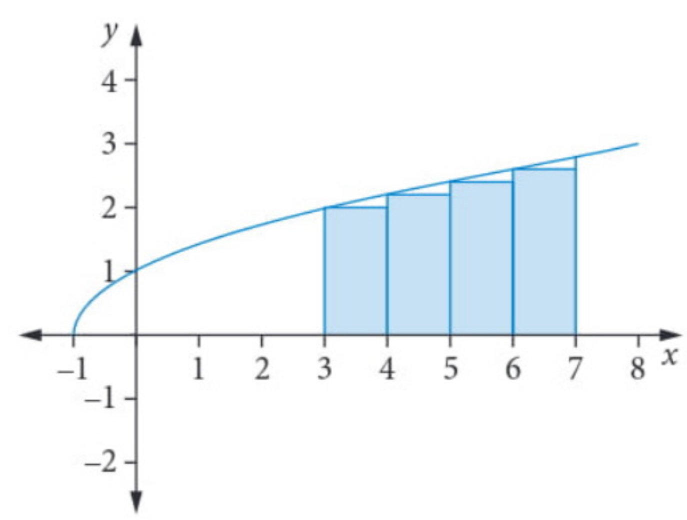
b $y = x^3$ between $x = 1$ and $x = 5$, using 4 rectangles.



c $y = 4x - x^2$ between $x = 1$ and $x = 4$, using 3 rectangles.

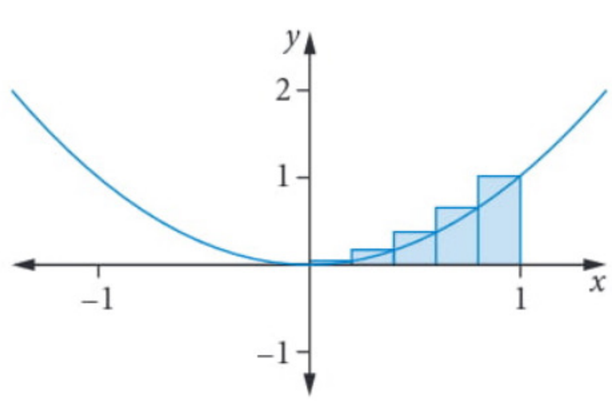


d $y = \sqrt{x+1}$ between $x = 3$ and $x = 7$, using 4 rectangles. Give your answer correct to two decimal places.

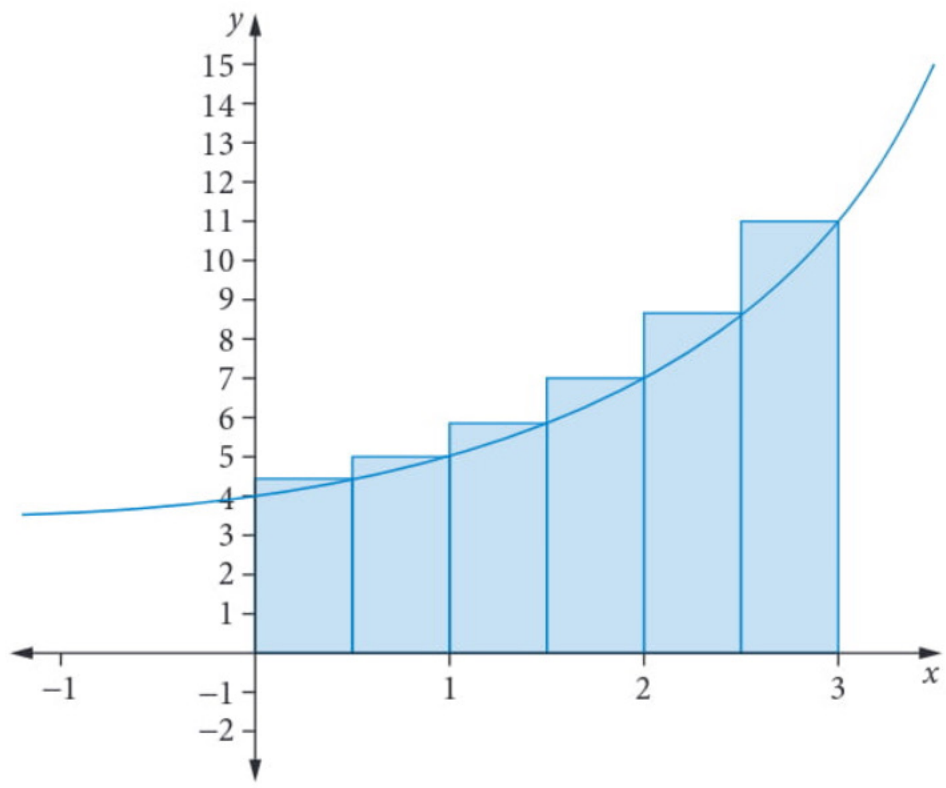


4 WORKED EXAMPLE 5 Use rectangles to find an underestimate and overestimate approximation to each area. Some rectangles have been drawn already.

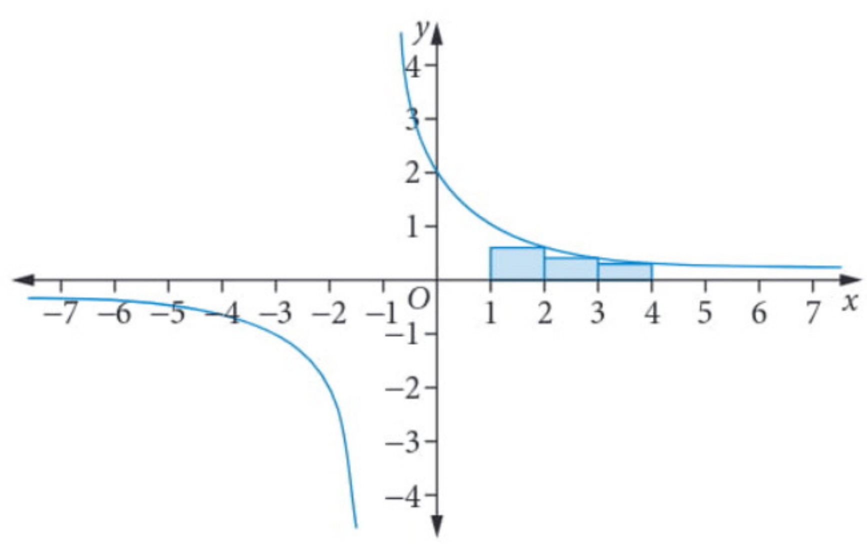
a $y = x^2$ between $x = 0$ and $x = 1$, using 5 rectangles. (Note: The first rectangle is very low, so doesn't show up well on this diagram.)



b $f(x) = 2^x + 3$ between $x = 0$ and $x = 3$, using 6 rectangles. Give your answer correct to two decimal places.



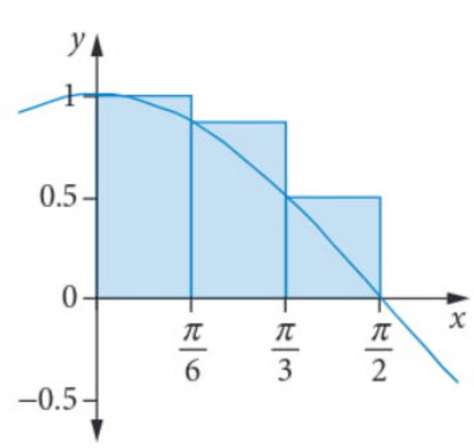
c $f(x) = \dfrac{2}{x+1}$ between $x = 1$ and $x = 4$, using 3 rectangles. Give your answer correct to two decimal places.



5 Using CAS 2 Consider the function $y = \dfrac{3}{2(x-1)}$ between $x = 2$ and $x = 4$. Use rectangles to find an approximation for the area described using underestimate and overestimate approximations, with the width of the rectangles being $\dfrac{1}{2}$ unit.

Calculator-free

6 (3 marks) Use the rectangles to show that the area under the curve $y = \cos(x)$ between $x = 0$ and $x = \dfrac{\pi}{2}$ is approximately $\dfrac{\pi}{12}\left[3 + \sqrt{3}\right]$ units^2.



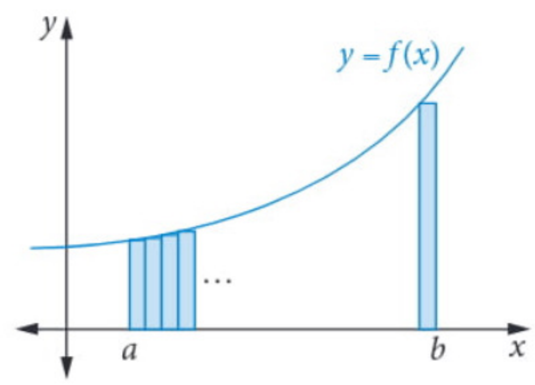
3.2

7 (4 marks) Find the approximate area under the curve $f(x) = 9 - x^2$ using rectangles as stated below.

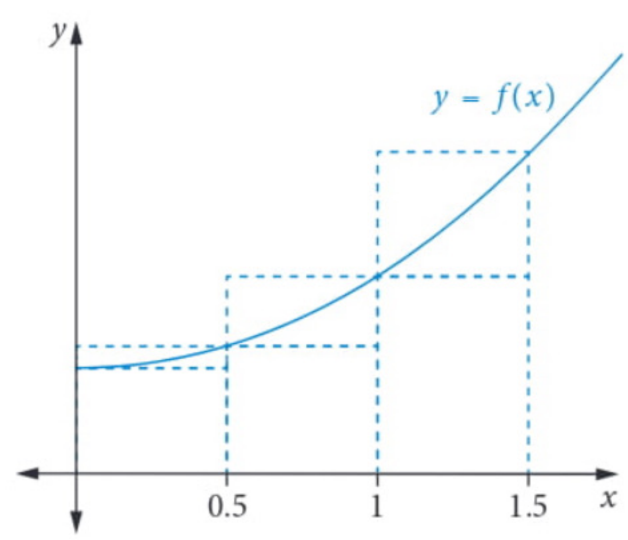
a between $x = 0$ and $x = 3$, with the width of each rectangle being 1 unit (2 marks)

b between $x = 0$ and $x = 2$, with the width of each rectangle being $\frac{1}{2}$ unit (2 marks)

8 (2 marks) An approximation to an area under a curve is to be found by summing the area of n rectangles of width h units that lie under the curve $y = f(x)$ between $x = a$ and $x = b$, as shown below. By considering the values of n and h, under what conditions will the approximation be most accurate?



9 © SCSA MM2017 Q9 (8 marks) Consider the function $f(x)$ shown graphed below. The table gives the value of the function at the given x values.



x	0	0.5	1	1.5
$f(x)$	20	21	24	29

a By considering the areas of the rectangles shown, demonstrate and explain why $32.5 < \int_0^{1.5} f(x)\,dx < 37$. (3 marks)

Consider the table of further values of $f(x)$ given below.

x	0	0.5	1	1.5	2	2.5	3
$f(x)$	20	21	24	29	36	45	56

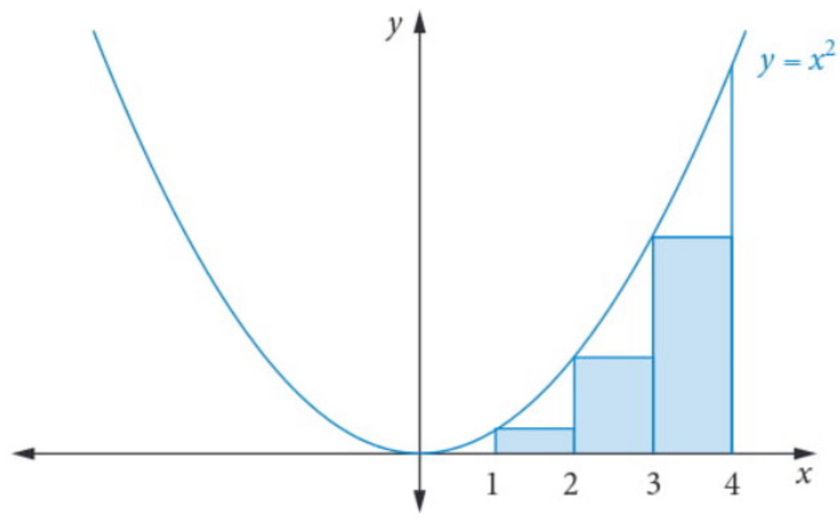
b Use the table values to determine the best estimate possible for $\int_1^3 f(x)\,dx$. (3 marks)

c State **two** ways in which you could determine a more accurate value for $\int_1^3 f(x)\,dx$. (2 marks)

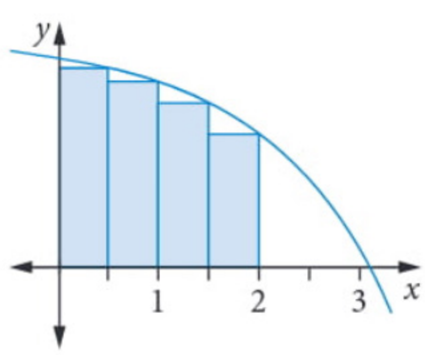
Calculator-assumed

10 (3 marks) Consider the graph of $f(x) = \sqrt{x}$. To find an approximation to the area of the region bounded by the graph of f, the x-axis and the line $x = 4$, four rectangles of equal width are drawn, and their total area calculated. By finding the mean of an underestimation and overestimation, what is this area correct to three decimal places?

11 (2 marks) Using the rectangles shown, would an underestimation of the approximate area under the curve $y = f(x)$ between $x = 1$ and $x = 4$ be found by evaluating $f(1) + f(2) + f(3)$ or $f(2) + f(3) + f(4)$? Explain.



12 (3 marks) Determine an approximation to the area under the curve $y = 10 - x^2$ between $x = 0$ and $x = 2$ using four rectangles as shown.



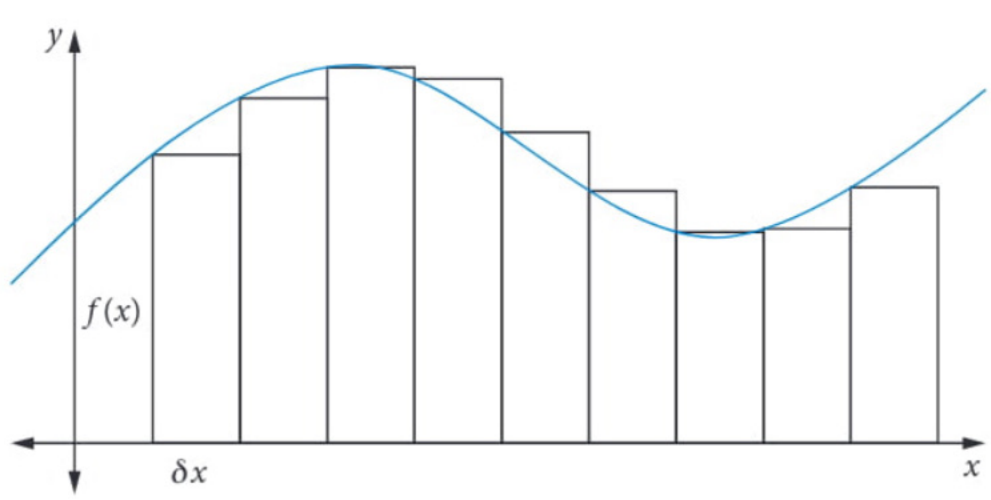
3.3 The definite integral and the fundamental theorem of calculus

Video playlist The definite integral and the fundamental theorem of calculus

Worksheet Definite integrals

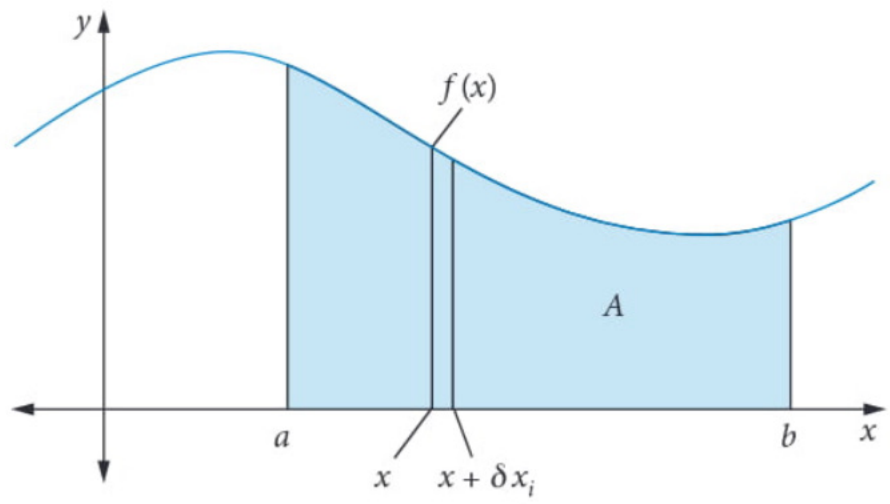
The definite integral

The area under a curve can be approximated by a sum of the areas of rectangles.



Each rectangle has the same width δx and different height $f(x)$, the y value of the function.

The area A under the curve $y = f(x)$ between $x = a$ and $x = b$ can be approximated by the sum of the areas of rectangles of width δx and height $f(x)$.



Algebraically, this is written as $A \approx \sum_{x=a}^{b} f(x)\delta x$ where $\sum_{x=a}^{b}$ means 'the sum of the expression for values of x between a and b'.

As more rectangles are used, the sum of the area of the vertical slices that fill the area gets closer and closer to the actual area.

> δx can also be written as Δx.
> δ is the lowercase Greek letter 'delta'.
> Δ is the capital Greek letter 'delta'.
> $\sum$ is the capital Greek letter 'sigma'.

Definite integrals

As more rectangles are used, their widths become smaller, so as δx approaches 0,

$A \approx \sum_{x=a}^{b} f(x)\delta x$ becomes the **definite integral** $A = \int_a^b f(x)\,dx$.

- A definite integral $\int_a^b f(x)\,dx$ with **limits** a and b has a value that is related to the area under a curve and is read as 'the integral of $f(x)$ between a and b with respect to x'.

  Thus, $\int_a^b f'(x)dx = f(b) - f(a)$.

- An **indefinite integral** $\int f(x)\,dx$ is an anti-derivative function and is read as 'the integral of $f(x)$ with respect to x'.

WORKED EXAMPLE 6 Calculating definite integrals

Evaluate each definite integral.

a $\int_0^3 5x^2\,dx$ **b** $\int_1^2 (x^3 + 4)\,dx$

Steps	Working
a 1 Integrate $5x^2$. **2** Substitute the limits of the integral $x = 3$ and $x = 0$ and subtract: $F(b) - F(a)$.	$\int_0^3 5x^2\,dx = \left[\frac{5x^3}{3}\right]_0^3 = \frac{5(3^3)}{3} - \frac{5(0^3)}{3}$ $= 45 - 0$ $= 45$ The '+ c' is not required when evaluating definite integrals because + c will cancel itself out in the subtraction.

b **1** Integrate $x^3 + 4$.

$$\int_1^2 (x^3 + 4)\,dx = \left[\frac{x^4}{4} + 4x\right]_1^2$$

2 Substitute the limits of the integral $x = 2$ and $x = 1$ and subtract: $F(b) - F(a)$.

$$= \left(\frac{2^4}{4} + 4(2)\right) - \left(\frac{1^4}{4} + 4(1)\right)$$

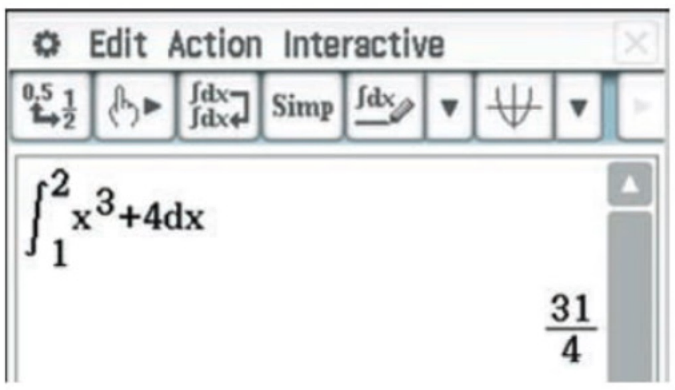
$$= 12 - \frac{17}{4}$$

$$= \frac{31}{4}$$

USING CAS 3 Definite integrals

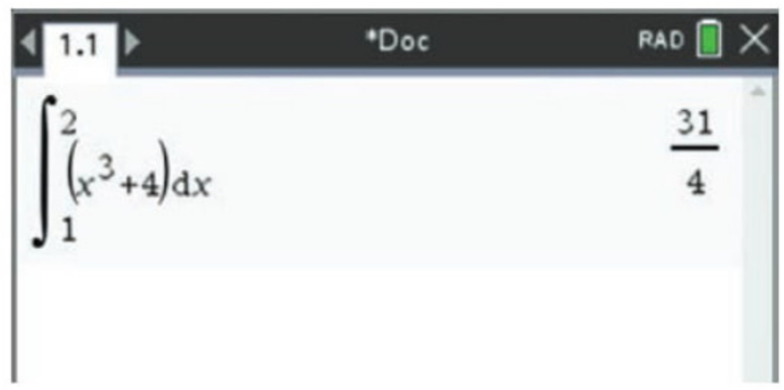
Evaluate the definite integral $\int_1^2 (x^3 + 4)dx$.

ClassPad



1 Highlight the given expression and tap **Interactive > Calculation > ∫**.
2 In the dialogue box, tap **Definite**.
3 Enter the lower and upper limits.

TI-Nspire



1 Press **menu > Calculus > Integral**.
2 Enter the lower and upper limits.
3 Enter the expression followed by **dx**.

$$\int_1^2 (x^3 + 4)\,dx = \frac{31}{4}$$

The fundamental theorem of calculus

The fundamental theorem of calculus is an important concept as it establishes a relationship between differentiation and integration. We begin by defining the **signed area** function $F(x) = \int_a^x f(t)dt$.

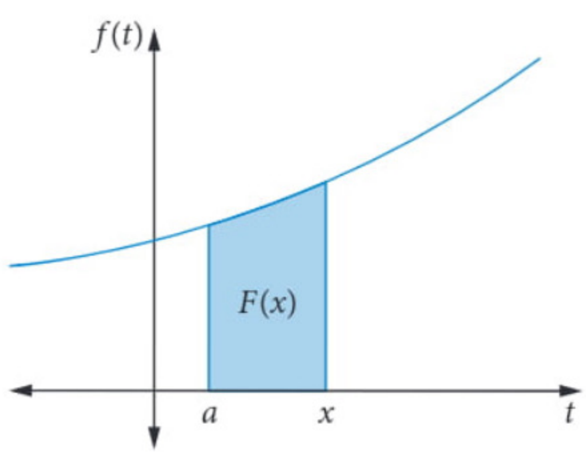


Figure 1

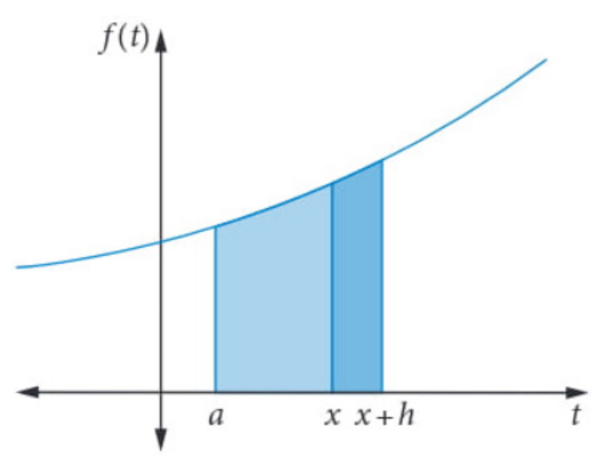


Figure 2

By definition: $F'(x) = \lim_{h\to 0} \dfrac{F(x+h) - F(x)}{h}$ (Figure 1)

Therefore, $F'(x) = \lim_{h\to 0} \dfrac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} = \lim_{h\to 0} \dfrac{\int_x^{x+h} f(t)\,dt}{h} \approx \lim_{h\to 0} \dfrac{hf(x)}{h} = \lim_{h\to 0} f(x) = f(x)$. (Figure 2)

Therefore, $F'(x) = \dfrac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$.

The fundamental theorem of calculus

The two parts of the fundamental theorem are

$$F'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x) \qquad \text{and} \qquad \int_a^b f'(x)\,dx = f(b) - f(a)$$

3.3

WORKED EXAMPLE 7 Applying the fundamental theorem of calculus

a Determine $\frac{d}{dx}\int_4^x t^2 + 3\,dt$.

b If $F(x) = \int_{-2}^x \frac{dt}{2t^2+1}$, determine $F'(x)$.

Steps	Working
a Apply the fundamental theorem to replace t with x.	$\frac{d}{dx}\int_4^x t^2 + 3\,dt = x^2 + 3$
b Apply the fundamental theorem to replace t with x.	$F'(x) = \frac{d}{dx}\int_{-2}^x \frac{dt}{2t^2+1} = \frac{1}{2x^2+1}$

Properties of the definite integral

1 $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	**Constant out:** a constant factor can be taken out of an integral.
2 $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	**Split terms:** a sum or difference of terms can be integrated separately.
3 If b is between a and c, then $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$	**Split limits:** the limits of an integral can be split.
4 $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$	**Swap limits:** reversing the order of the limits changes the sign of the definite integral.
5 $\int_a^a f(x)\,dx = 0$	**Same limits:** gives $F(a) - F(a) = 0$.

The next two examples illustrate some exam-type questions where we are not given the equation of the function.

WORKED EXAMPLE 8	**Properties of definite integrals**
If $\int_0^2 g(x)\,dx = 3$, evaluate $\int_2^0 (g(x)+2x)\,dx$.	
Steps	**Working**
1 Simplify using Property 2: split terms.	$\int_2^0 (g(x)+2x)\,dx = \int_2^0 g(x)\,dx + \int_2^0 (2x)\,dx$
2 Simplify using Property 4: swap limits.	$= -\int_0^2 g(x)\,dx - \int_0^2 (2x)\,dx$
3 Substitute $\int_0^2 g(x)\,dx = 3$.	$= -3 - \int_0^2 (2x)\,dx$
4 Evaluate and simplify.	$= -3 - \left[x^2\right]_0^2$ $= -3 - (2^2 - 0^2)$ $= -7$

WORKED EXAMPLE 9	**Properties of definite integrals**
If $\int_1^2 f(x)\,dx = 10$, evaluate $\int_1^2 3(f(x)-5)\,dx$.	Note carefully where the common factor is placed.
Steps	**Working**
1 Simplify using Property 1: constant out.	$\int_1^2 3(f(x)-5)\,dx = 3\int_1^2 (f(x)-5)\,dx$
2 Simplify using Property 2: split terms.	$= 3\left[\int_1^2 f(x)\,dx - \int_1^2 5\,dx\right]$
3 Substitute $\int_1^2 f(x)\,dx = 10$.	$= 3\left[10 - \int_1^2 5\,dx\right]$
4 Evaluate and simplify.	$= 30 - 3\left[5x\right]_1^2$ $= 30 - 3(10-5)$ $= 15$

EXERCISE 3.3 The definite integral and the fundamental theorem of calculus

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3.3

Recap

1 Use rectangles of width $\frac{1}{2}$ unit to find the area under the curve $y = x^2 + 1$ from $x = 0$ to $x = 2$.

2 When using rectangles to approximate the area under a curve, what effect will reducing the width of the rectangles have on the accuracy of the approximation? Explain.

Mastery

3 WORKED EXAMPLE 6 Evaluate each definite integral.

a $\int_1^3 4x\,dx$ **b** $\int_0^2 7x^6\,dx$ **c** $\int_1^2 4x^3\,dx$

d $\int_2^3 (2x-1)^2\,dx$ **e** $\int_0^4 (x+2)^{-2}\,dx$ **f** $\int_0^1 (x^3 - 3x^2 + 1)\,dx$

4 Using CAS 3 Evaluate

a $\int_0^2 (2x^2 + 4x)\,dx$ **b** $\int_2^0 (2x^2 + 4x)\,dx$

5 WORKED EXAMPLE 7

a Determine $\frac{d}{dx}\int_0^x 2t^2 + t - 4\,dt$. **b** If $F(x) = \int_{-2}^x \frac{3\,dt}{t^2 - 1}$, determine $F'(x)$.

6 WORKED EXAMPLE 8 If $\int_1^3 f(x)\,dx = 3$, evaluate $\int_1^3 -5f(x)\,dx$.

7 WORKED EXAMPLE 9 If $\int_0^2 g(x)\,dx = 5$, evaluate $\int_0^2 4 - 3g(x)\,dx$.

Calculator-free

8 (10 marks) Evaluate each definite integral.

a $\int_0^2 \frac{x^2}{2}\,dx$ **b** $\int_{-1}^1 (3x^2 + 4x)\,dx$ **c** $\int_{-1}^2 (x^2 + 1)\,dx$

d $\int_{-2}^3 (4x^3 - 3)\,dx$ **e** $\int_{-1}^0 (x^2 + 3x + 5)\,dx$

9 (4 marks) Determine

a $\frac{d}{dx}\int_0^x \sqrt{t - \pi}\,dt$ **b** $\frac{d}{dx}\int_x^0 -2t^2 + t\,dt$

10 (2 marks) If $F(x) = \frac{3x^2}{2} + 2\int_0^x 1 - 2t^2\,dt$, determine $F'(x)$.

11 (3 marks) Evaluate $\int_1^4 (\sqrt{x} + 1)\,dx$.

12 (4 marks) Write each expression as one integral. Do not evaluate.

a $\int_0^1 x^2\,dx + \int_1^5 x^2\,dx$

b $\int_1^4 (x+1)\,dx + \int_4^7 (x+1)\,dx$

c $\int_{-2}^0 (x^3 - x - 1)\,dx + \int_0^2 (x^3 - x - 1)\,dx$

d $\int_0^2 (2x+1)\,dx + \int_2^3 (2x+1)\,dx$

Calculator-assumed

13 (2 marks) Evaluate $\int_0^1 (x^2 - x^3)\,dx$.

14 (2 marks) Evaluate $\int_1^2 \left(3x^2 + \frac{4}{x^2}\right) dx$.

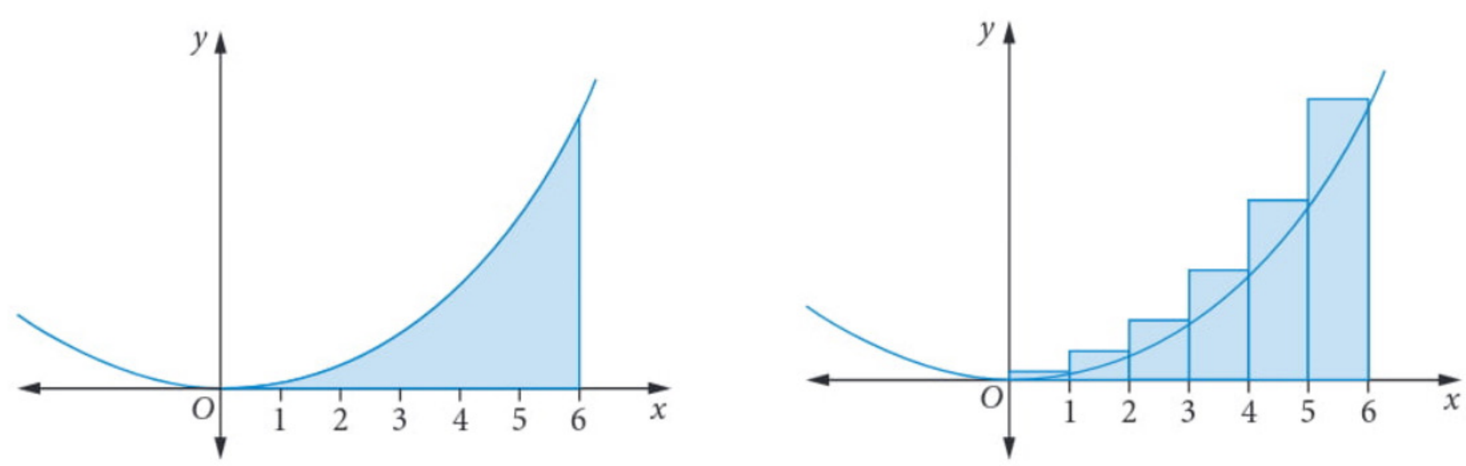
15 (3 marks) The value of the integral of $f(x) = \frac{1}{a-1}(x^2 - 2x)$ over the interval $[1, a]$ is $\frac{13}{3}$. Determine the value of a.

16 (2 marks) If $\int_1^3 f(x)\,dx = 5$, determine the value of $\int_1^3 (2f(x) - 3)\,dx$.

17 (2 marks) If $\int_1^{12} g(x)\,dx = 5$ and $\int_{12}^5 g(x)\,dx = -6$, determine the value of $\int_1^5 g(x)\,dx$.

18 (2 marks) If $F(x)$ is an anti-derivative of $f(x)$ and $F(4) = -6$, then explain why $F(8)$ is equal to $\int_4^8 (-6 + f(x))\,dx$.

19 (2 marks) A part of the graph of $f(x) = x^2$ is shown. Zoe finds the approximate area of the shaded region by drawing rectangles as shown in the second diagram.



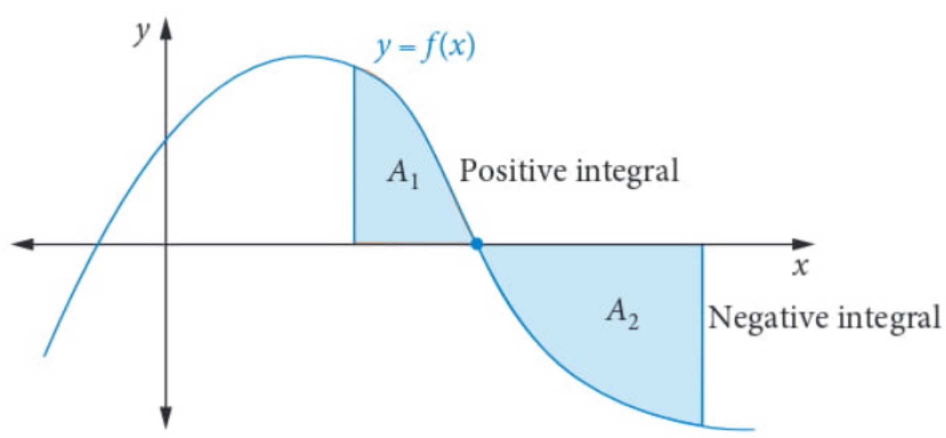
Zoe's approximation is p% more than the exact value of the area. Determine if p is to closer to 20, 25 or 30.

3.4 Area under a curve

Area under a curve

The definite integral $\int_a^b f(x)\,dx$ gives the signed area enclosed by the graph of $y = f(x)$ and the x-axis between $x = a$ and $x = b$.

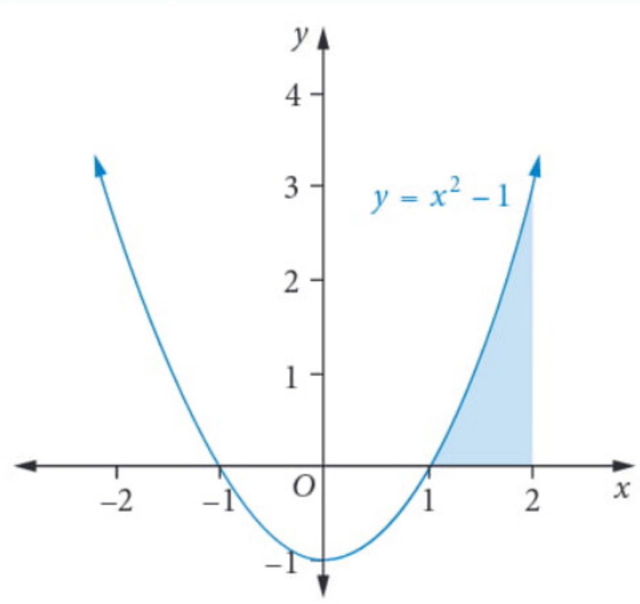
- If the section of a graph is above the x-axis, then the definite integral over that section gives the area under the curve.
- If the section of a graph is below the x-axis, then the definite integral over that section gives the area above the curve and its value is negative.
- If the graph is partly above and partly below the x-axis, then we split the graph and the integral to find the required area.



Video playlist
Area under a curve

WORKED EXAMPLE 10 **Finding the area under a curve**

Find the area enclosed by the graph of $y = x^2 - 1$ and the x-axis between $x = 1$ and $x = 2$.

Steps	Working
1 Sketch the graph showing the positive area required.	
2 Write the integral required to find the area.	$\int_1^2 (x^2 - 1)\,dx$
3 Evaluate the area.	$\int_1^2 (x^2 - 1)\,dx = \left[\frac{x^3}{3} - x\right]_1^2$ $= \left(\frac{2^3}{3} - 2\right) - \left(\frac{1^3}{3} - 1\right)$ $= \frac{4}{3}$ units^2

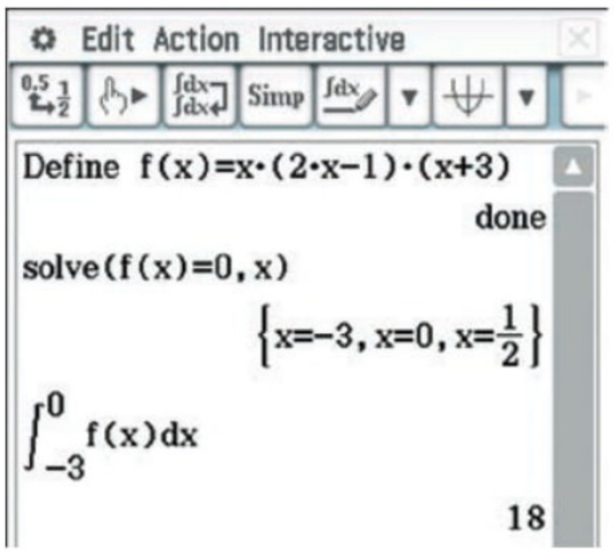
Exam hack

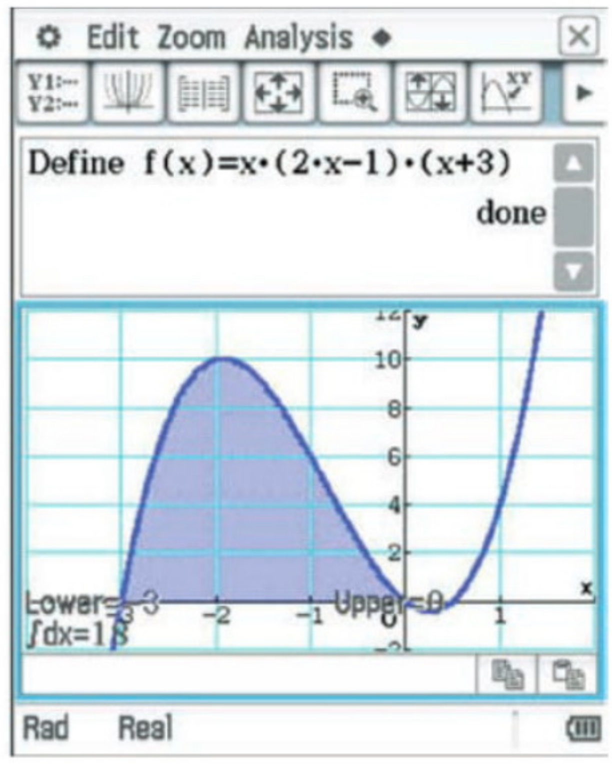
Always sketch a graph first to see if we are looking for a positive or a negative area or a mixture of both.

USING CAS 4 Area under a curve

Find the area under the curve $y = x(2x - 1)(x + 3)$ enclosed by the graph and the x-axis where the area is positive.

ClassPad





1 Define **f(x)** as shown above.

2 Solve **f(x) = 0** to determine the x-intercepts.

3 Find the definite integral of **f(x)** from **–3** to **0**.

4 The value of the positive area will be displayed.

5 To confirm this result, graph **f(x)**.

6 Adjust the window settings to suit.

7 Tap **Analysis > G-Solve > Integral > ∫dx.**

8 Enter **–3**.

9 A dialogue box will appear with **–3** in the **Lower:** field.

10 Enter **0** in the **Upper:** field.

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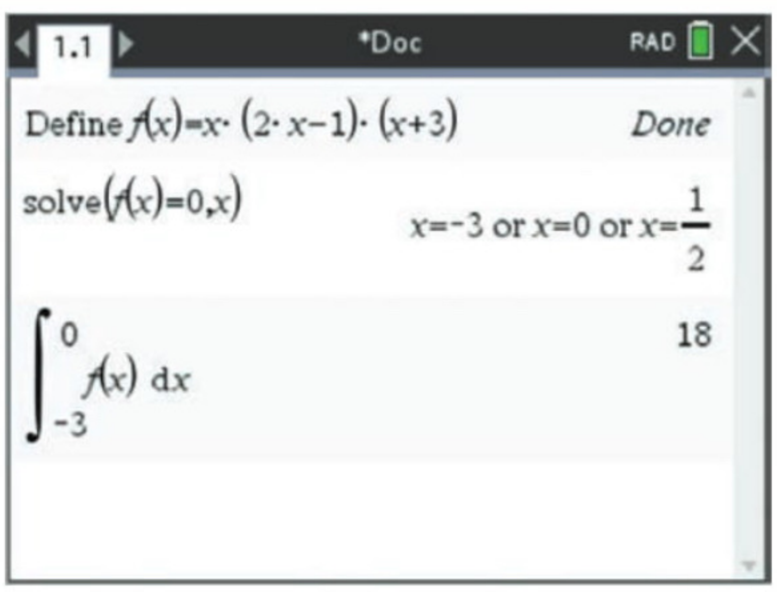
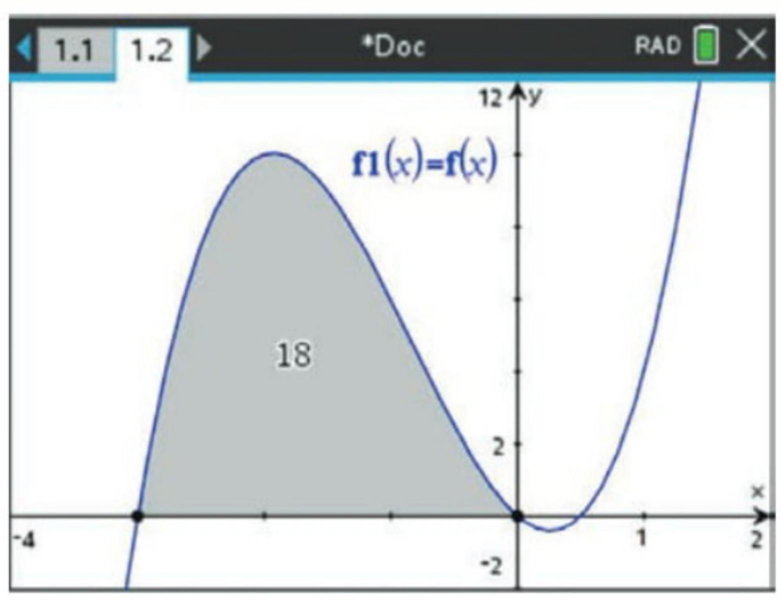




1 Define **f(x)** as shown above.

2 Solve **f(x) = 0** to determine the x-intercepts.

3 Find the definite integral of **f(x)** from **–3** to **0**.

4 The value of the positive area will be displayed.

5 To confirm this result, graph **f(x)**.

6 Adjust the window settings to suit.

7 Press **menu > Analyze Graph > Integral**.

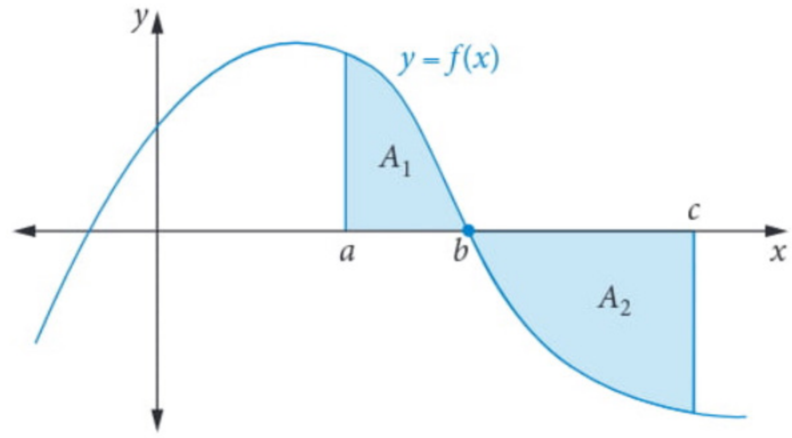
8 When prompted for the **lower bound**, click on **–3** on the x-axis.

9 When prompted for the **upper bound**, click on **0** on the x-axis.

The area under the curve is 18 units^2.

Areas above and below curves

The definite integral is negative for areas below the x-axis. When finding areas using integration, always check if the graph is below the x-axis.



In this diagram, $A_1 = \int_a^b f(x)\,dx$ is positive, whereas $A_2 = \int_b^c f(x)\,dx$ is negative.

So the area of A_1 is $\int_a^b f(x)\,dx$, whereas the area of A_2 is $-\int_b^c f(x)\,dx$ (changing the sign makes a negative area become positive).

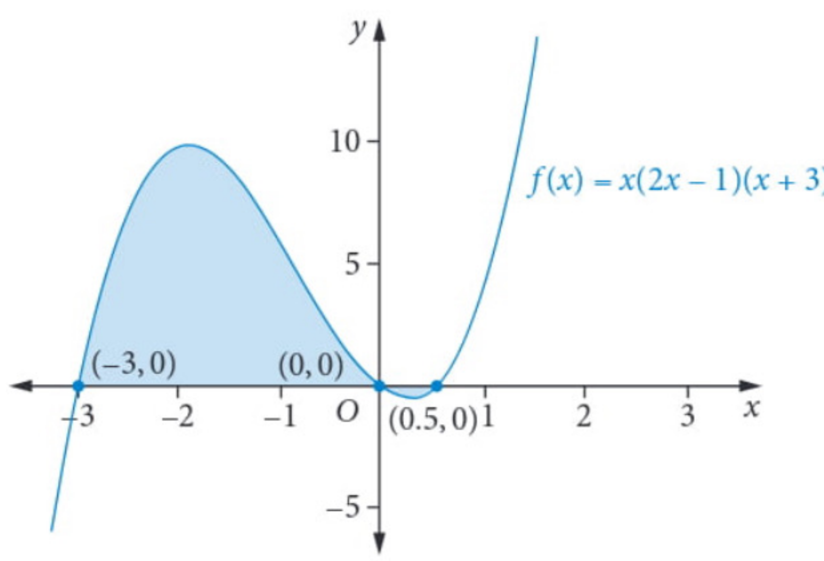
So the sum of the areas of A_1 and A_2 is $\int_a^b f(x)\,dx - \int_b^c f(x)\,dx$.

An alternative method for dealing with the area below the axis is to use the property of definite integrals of **swapping limits**.

In this case, the sum of the areas of A_1 and A_2 is $\int_a^b f(x)\,dx + \int_c^b f(x)\,dx$, where we swap the limits of b and c.

Worksheet
Calculating physical areas

For example, we can calculate the total shaded area in this graph in two different ways.

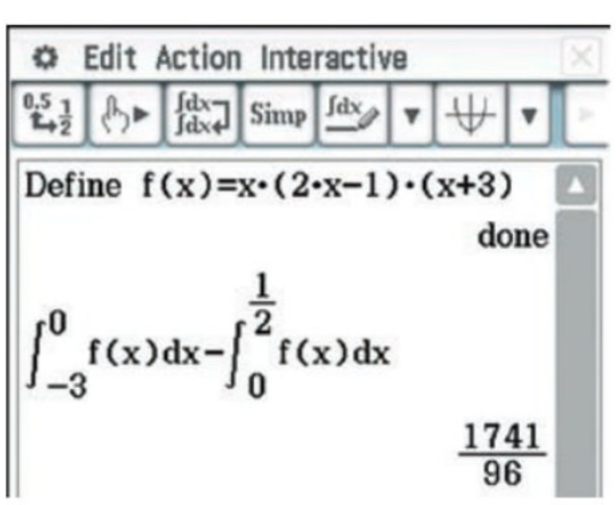


Method 1: Subtract the integral that would give a negative area.

$$\text{Area} = \int_{-3}^{0}(x(2x-1)(x+3))\,dx - \int_{0}^{\frac{1}{2}}(x(2x-1)(x+3))\,dx$$

$$= \frac{1741}{96}\text{ units}^2$$

ClassPad



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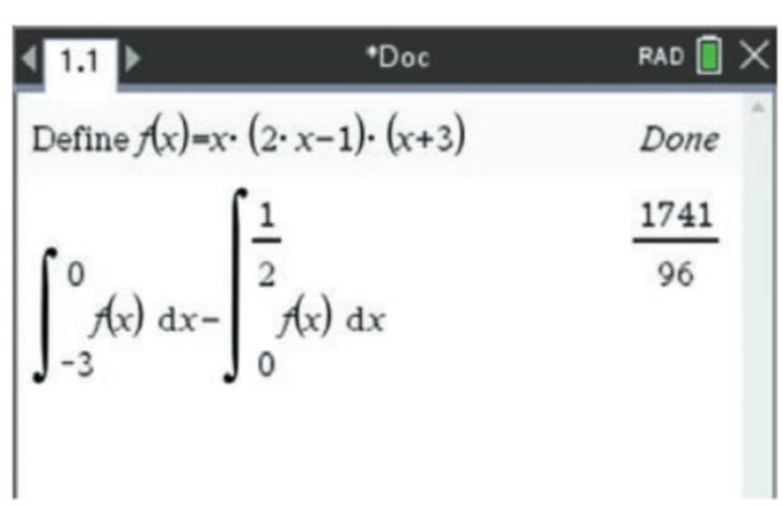


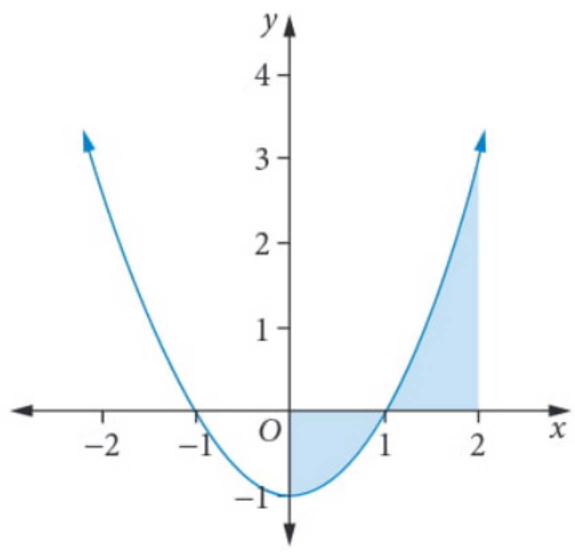
Method 2: Swap the limits of the integral that would give a negative area.

$$\text{Area} = \int_{-3}^{0}(x(2x-1)(x+3))\,dx + \int_{\frac{1}{2}}^{0}(x(2x-1)(x+3))\,dx$$

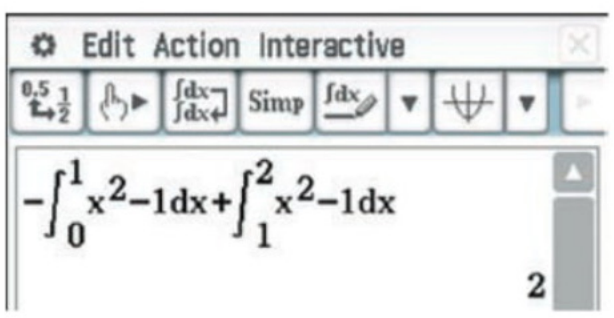
$$= \frac{1741}{96}\text{ units}^2$$

WORKED EXAMPLE 11 Areas above and below curves

Find the area enclosed by the graph of $y = x^2 - 1$ and the x-axis between $x = 0$ and $x = 2$.

Steps	Working
1 Sketch the graph, showing the area required.	
2 Find the x-intercept that splits the area into two regions, above and below the x-axis.	$y = x^2 - 1 = 0$ $x^2 = 1$ $x = \pm 1$ $x = 1$ is where the area is split.
3 Write the integral required to find the split area.	Area $= -\int_0^1 (x^2 - 1)\,dx + \int_1^2 (x^2 - 1)\,dx$
4 Evaluate the integrals.	Area = 2 units^2

ClassPad



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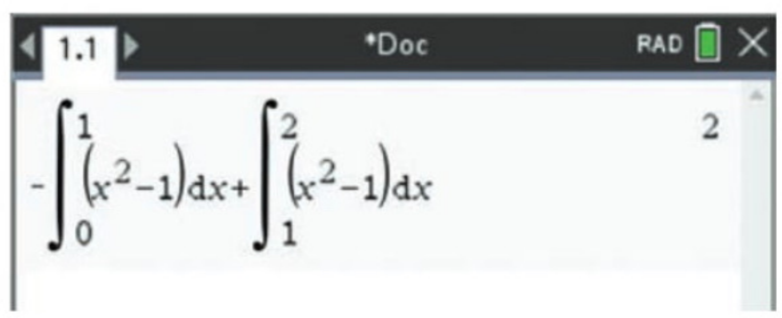


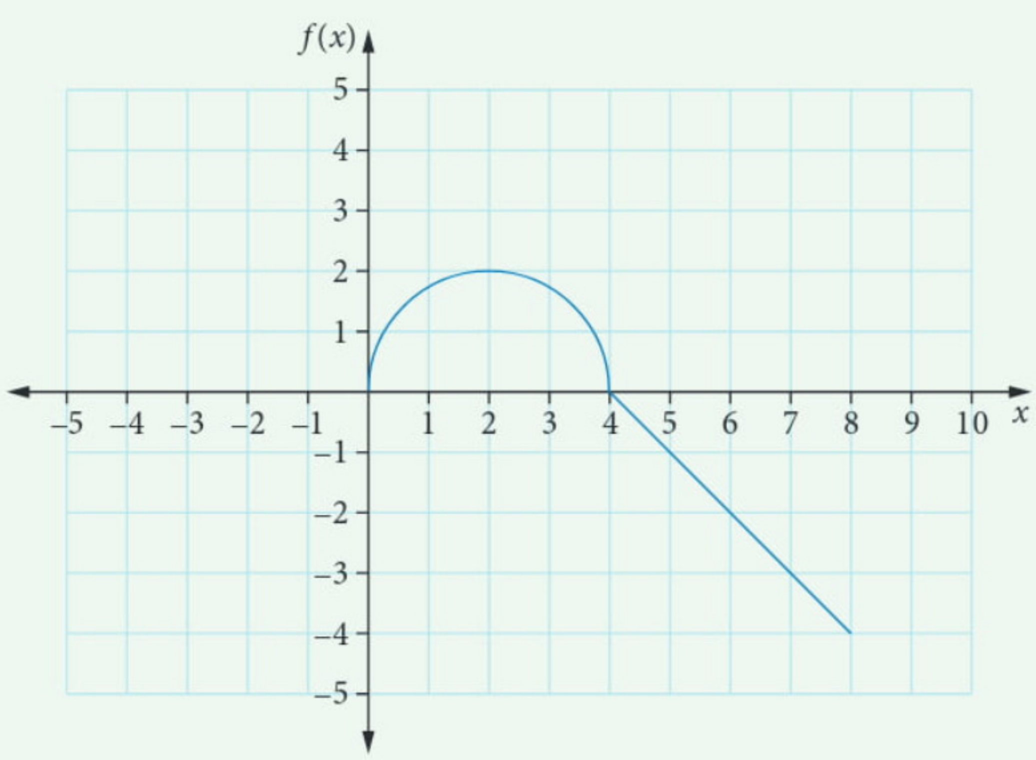
Sometimes we are not given the function but are given information about the areas under the curve. This information can be used to evaluate definite integrals.

WORKED EXAMPLE 12 Using areas under the curve to determine integrals

Given the graph of $f(x)$, determine the exact value of

a $\int_0^2 f(x)\,dx$

b $\int_0^6 f(x)\,dx$



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Steps	Working
a Note that the required area is quarter of a circle, with a radius of 2.	$\int_0^2 f(x)\,dx = \frac{\pi(2)^2}{4} = \pi$ As we do not know the function, we use the area formula to get the answer.
b **1** Determine the area of the semicircle.	$\int_0^4 f(x)\,dx = \frac{\pi(2)^2}{2} = 2\pi$ (or double your answer from part **a**)
2 Determine the area of the triangle under the axis (between 4 and 6).	$\int_4^6 f(x)\,dx = \frac{2\times 2}{2} = 2$ But as the area is under the curve, the answer is −2.
3 Add your answers together.	$\int_0^6 f(x)\,dx = 2\pi - 2$

Exam hack

Although we are using areas to calculate the answers, the question did not mention areas, so do not place units^2 on your answer.

EXERCISE 3.4 Area under a curve

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Recap

1 Evaluate $\int_1^2 (x^3 - 2x)dx$.

2 If $\int_0^5 g(x)\,dx = 20$ and $\int_0^5 (2g(x) + ax)\,dx = 90$, determine the value of a.

Mastery

3 WORKED EXAMPLE 10 Find the area between the x-axis and the graph of each function.

a $y = x^2 - 5x + 8$ from $x = 1$ to $x = 4$

b $f(x) = 15 + 8x - 6x^2$ between $x = -1$ and $x = 2$

c $y = 4x^3 - 3x^2 + 6x - 2$ between $x = 1$ and $x = 3$

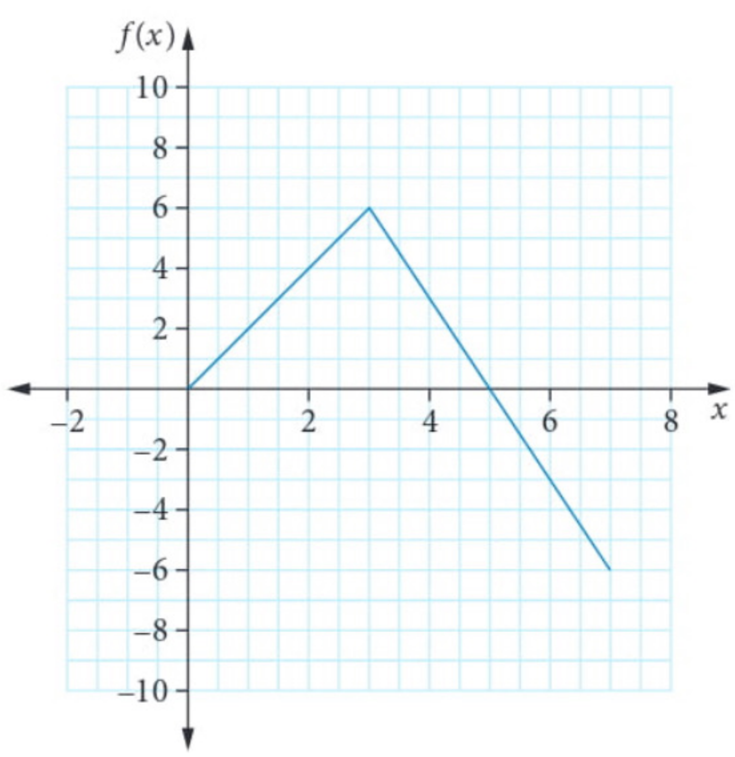
4 Using CAS 4 Determine the area between the graph of $y = x^3 - 2x^2 - 11x + 12$ and the x-axis from $x = 1$ to $x = 4$.

5 WORKED EXAMPLE 11 Determine the area between the graph of $y = x^3 - 2x^2 - 11x + 12$ and the x-axis from $x = -3$ to $x = 4$.

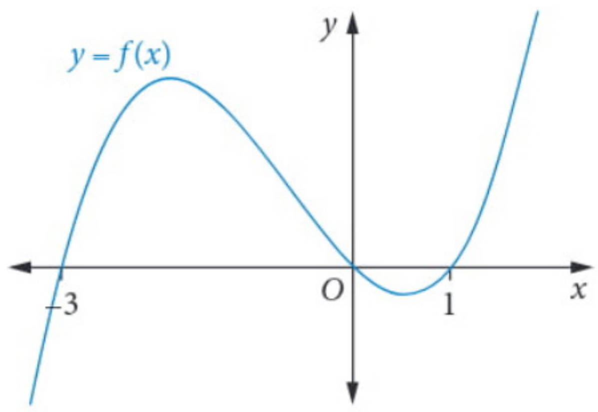
6 WORKED EXAMPLE 12 Given the graph of $f(x)$, determine the exact value of

a $\int_0^3 f(x)\,dx$

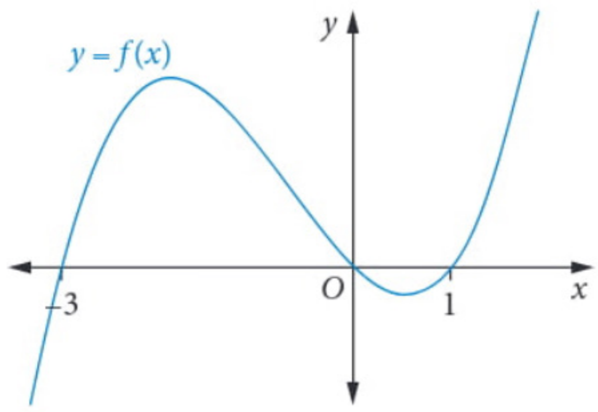
b $\int_0^7 f(x)\,dx$



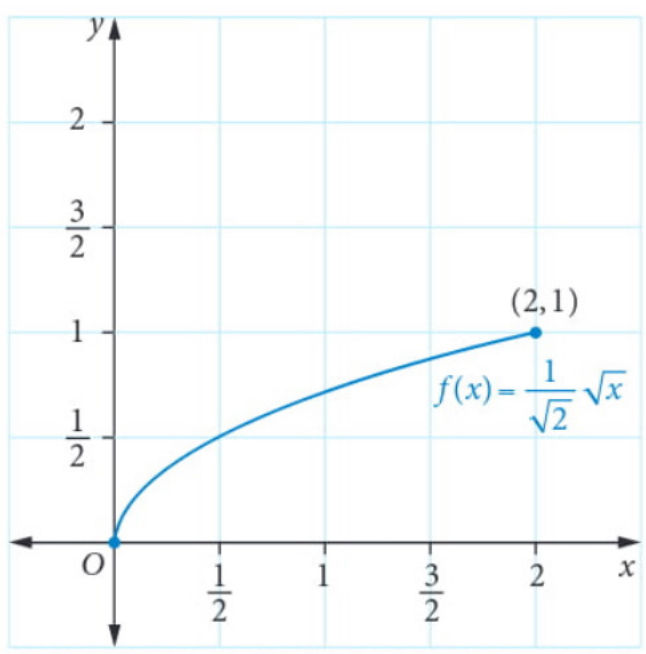
7 (2 marks) Write an integral expression that would determine the area between the graph of $f(x)$, the x-axis and the lines $x = -2$ and $x = -1$.



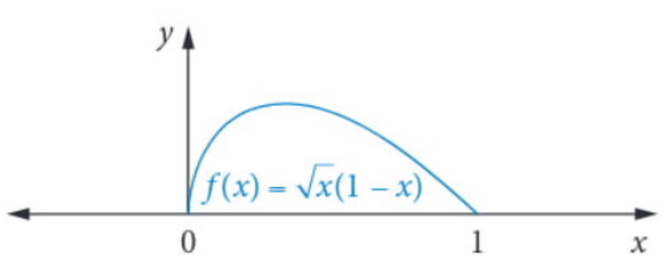
8 (2 marks) Write an integral expression that would determine the area between the graph of $f(x)$, the x-axis and the lines $x = -3$ and $x = 1$.



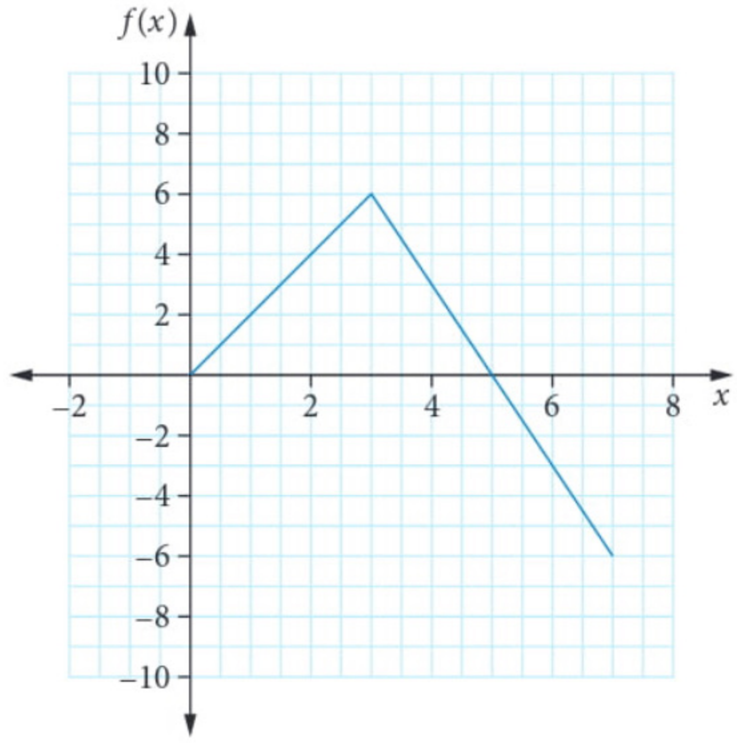
9 (3 marks) Find the area of the region bounded by the function $f(x) = \frac{1}{\sqrt{2}}\sqrt{x}$, the x-axis and the line $x = 2$.



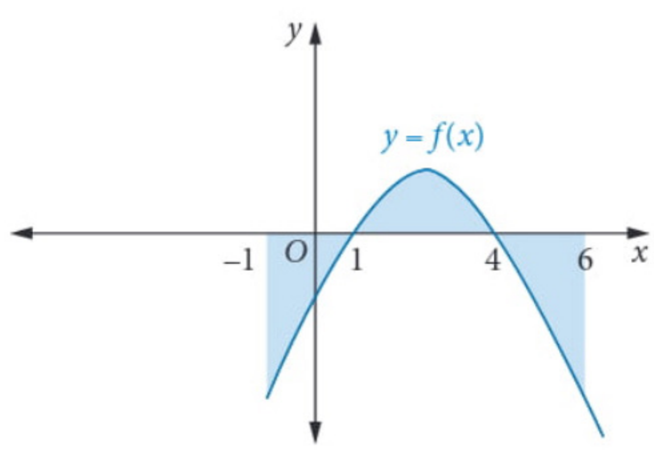
10 (3 marks) Part of the graph $f(x) = \sqrt{x}(1 - x)$ is shown below. Calculate the area between the graph of f and the x-axis.



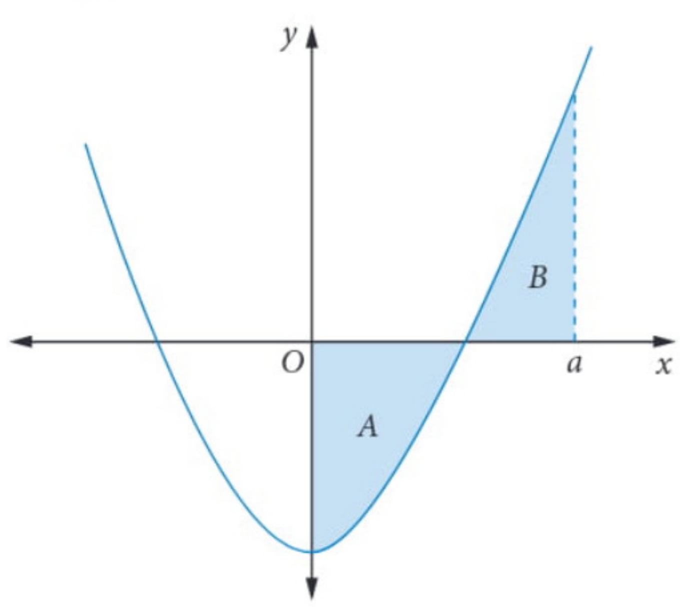
11 (4 marks) Given the graph of $f(x)$, determine the exact value of $\int_2^6 f(x)\, dx$.



12 (2 marks) Write an integral expression that would determine the total area of the shaded regions in the diagram.



13 (3 marks) A part of the graph of $g(x) = x^2 - 4$ is shown below.



If the area of the region marked A is the same as the area of the region marked B, determine the exact value of a.

14 (3 marks) Determine the area under the curve $y = 2(x + 1)^3$ from $x = -1$ to $x = 2$.

3.5 Areas between curves

Video playlist
Areas between curves

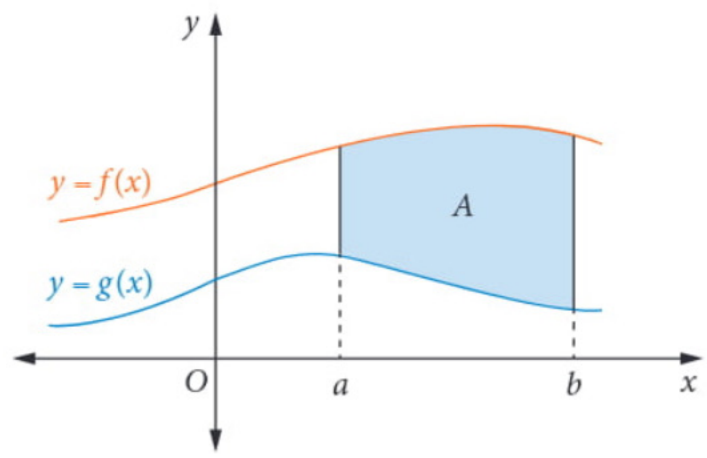
Worksheets
Calculating areas between curves

Areas between curves 1

Areas between curves 2

The area enclosed between two curves can be calculated as the difference between the areas under the two functions. In the diagram, $f(x) > g(x)$, so the shaded area can be found by subtracting areas:

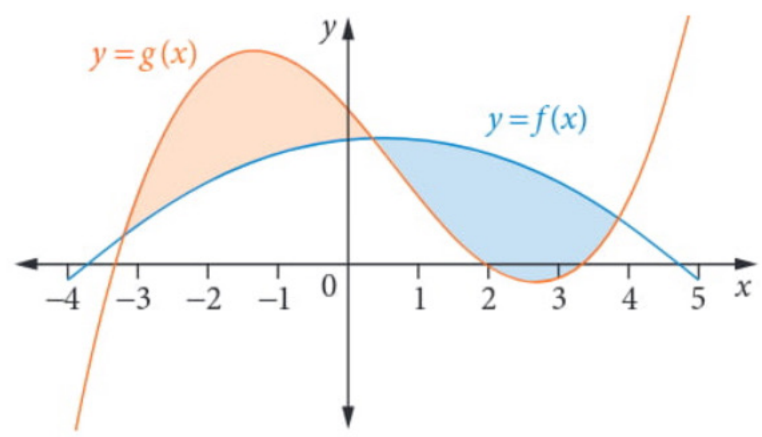
$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$$



Areas between curves

$$\text{Area} = \int_a^b (\text{upper} - \text{lower})\,dx = \int_a^b [f(x) - g(x)]\,dx$$

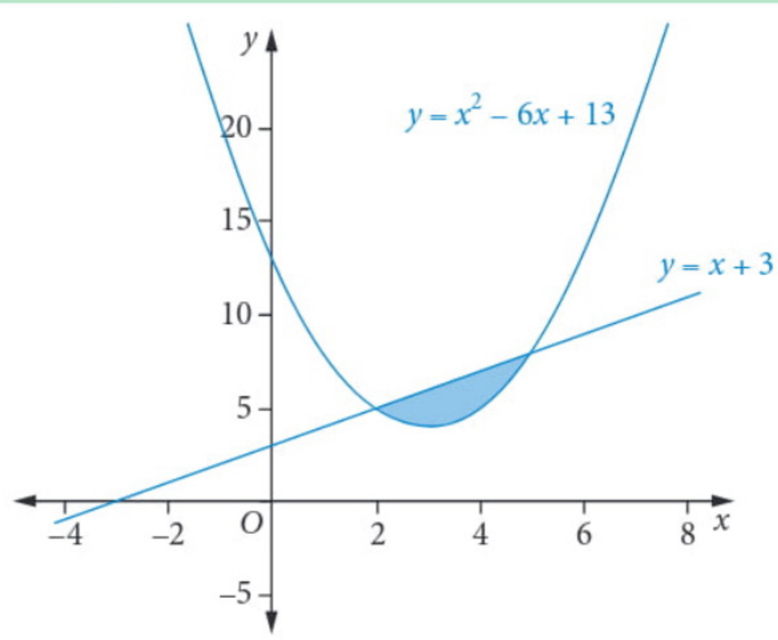
For curves that intersect, we need to find their point(s) of intersection and again using properties of definite integrals, split the integral limits, taking note of which function is greater. In the diagram, $g(x)$ is higher on the left, but $f(x)$ is higher on the right.



Note that when finding the area between two curves, the position of the area does not matter. That is, it does not matter if part of the area is below the axes.

WORKED EXAMPLE 13 Finding the area between curves

Find the area enclosed by the graphs of $y = x^2 - 6x + 13$ and $y = x + 3$.

Steps	Working
1 Sketch the graphs of $y = x^2 - 6x + 13$ and $y = x + 3$. Always draw a sketch first to determine which is the upper function and which is the lower function.	
2 Solve simultaneously to find the points of intersection.	$x^2 - 6x + 13 = x + 3$ $x^2 - 7x + 10 = 0$ $\therefore\ (x - 2)(x - 5) = 0$ $x = 2, x = 5$
3 Between $x = 2$ and $x = 5$, the line $y = x + 3$ is the upper function. Set up the integral using $\int_a^b (\text{upper} - \text{lower})\,dx$ and simplify the terms.	$\text{Area} = \int_2^5 ((x + 3) - (x^2 - 6x + 13))\,dx$ $= \int_2^5 (-x^2 + 7x - 10)\,dx$

4 Integrate and evaluate.

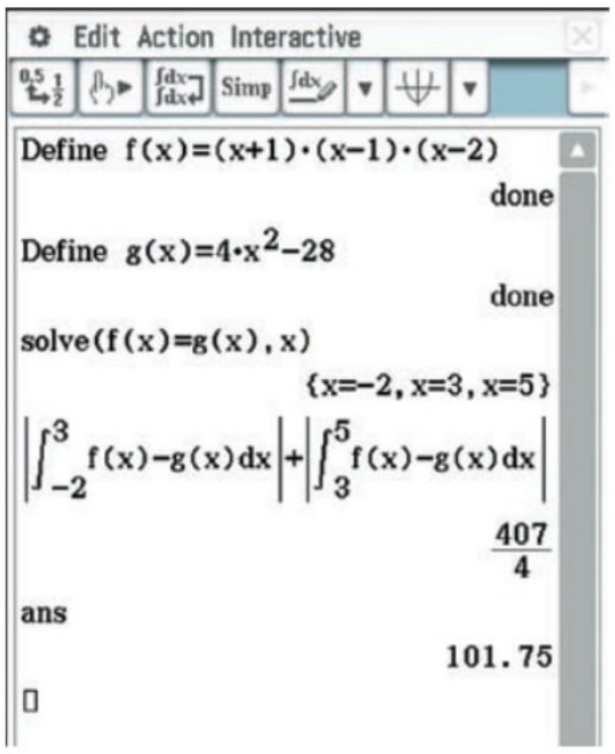
$$A = \left[-\frac{x^3}{3} + \frac{7x^2}{2} - 10x\right]_2^5$$
$$= \left(-\frac{5^3}{3} + \frac{7(5)^2}{2} - 10(5)\right) - \left(-\frac{2^3}{3} + \frac{7(2)^2}{2} - 10(2)\right)$$
$$= \frac{9}{2}$$
$$\therefore \text{ area} = \frac{9}{2} \text{ units}^2$$

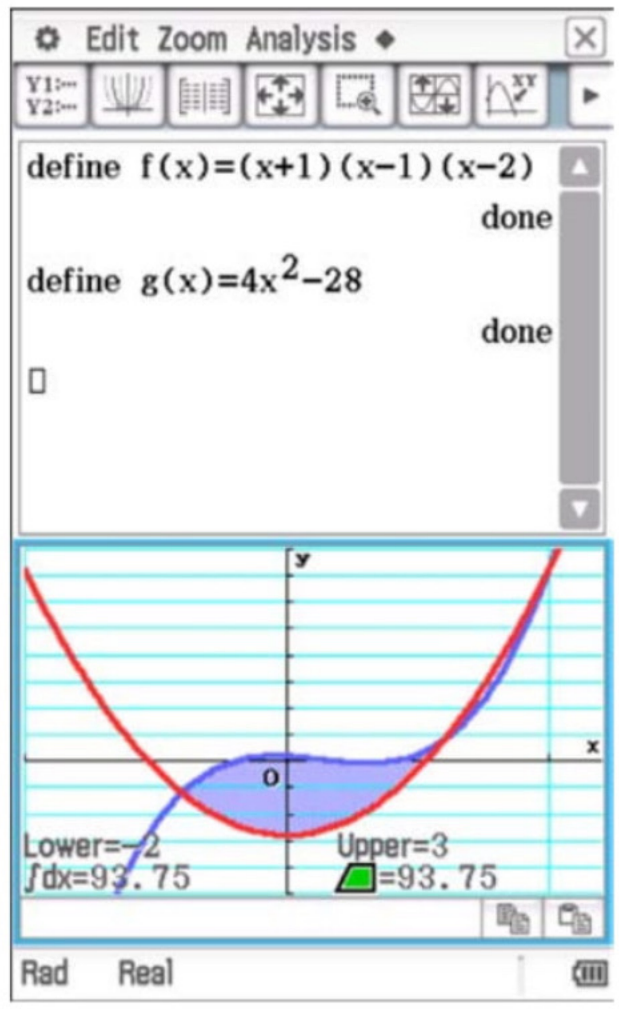
Areas of integration calculations are much easier to perform using CAS.

USING CAS 5 Finding the area between curves

Find the area between the curves $f(x) = (x + 1)(x - 1)(x - 2)$ and $g(x) = 4x^2 - 28$.

ClassPad





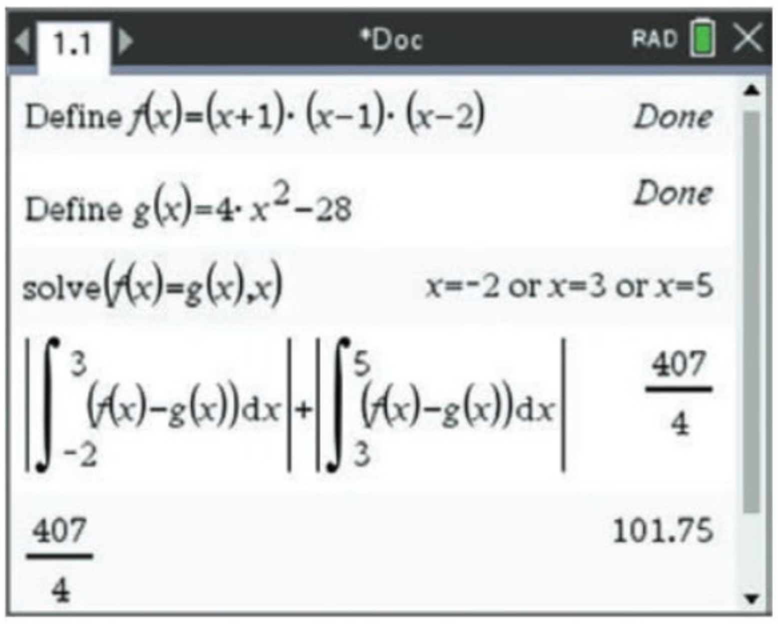
1. Define **f(x)** and **g(x)**, as shown above.
2. Solve **f(x) = g(x)** to determine the x values of the points of intersection.

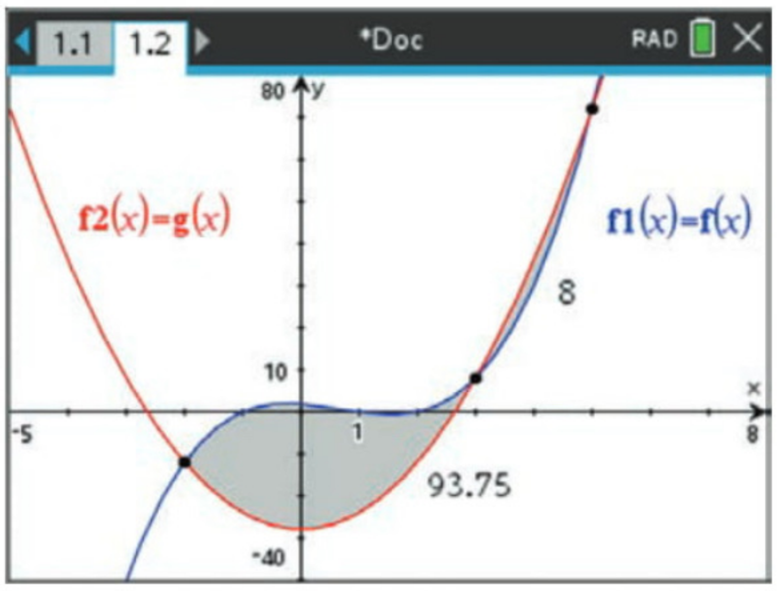
> The absolute value brackets '||' around an expression make its value always positive whether it is positive or negative.

3. From the **Keyboard**, tap **Math1** or **Math2** to insert the absolute value template **| |**.
4. Find the definite integral of **f(x)–g(x)** from **–2** to **3**.
5. Add the absolute value of the definite integral of **f(x)–g(x)** from **3** to **5**.
6. The sum of the areas between the curves will be displayed.
7. To confirm this result, graph **f(x)** and **g(x)**.
8. Adjust the window settings to suit.
9. Tap **Analysis > G-Solve > Integral > ∫dx Intersection**.
10. With the cursor on the first point of intersection, press **EXE**.
11. Press the right arrow to jump to the second point of intersection, press **EXE**.
12. The area between the first two points of intersection will be displayed.
13. Repeat to find the area between the second and third points of intersection.
14. Add the two values to find the total area.

Total area = 101.75 units^2

TI-Nspire





1 Define **f(x)** and **g(x)** as shown above.

2 Solve **f(x) = g(x)** to determine the x values of the points of intersection.

3 Press the **template** key and insert the absolute value template **| |**.

> The absolute value brackets '||' around an expression make its value always positive whether it is positive or negative.

4 Find the definite integral of **f(x)−g(x)** from **−2** to **3**.

5 Add the absolute value of the definite integral of **f(x)−g(x)** from **3** to **5**.

6 The sum of the areas between the curves will be displayed.

7 To confirm this result, graph **f(x)** and **g(x)**.

8 Adjust the window settings to suit.

9 Press **menu > Analyse Graph > Bounded Area**.

10 When prompted for the **lower bound**, click on first point of intersection on the left.

11 When prompted for the **upper bound**, click the second point of intersection.

12 Repeat to find the area between the second and third points of intersection.

13 Add the two values to find the total area.

Total area = 101.75 units^2

EXERCISE 3.5 Areas between curves

ANSWERS p. 393

Recap

1 The area enclosed by the graph of $f(x) = -x^2 + x$ and the x-axis between $x = -1$ to $x = 2$ can be expressed as

A $\int_{-1}^{2} f(x)\,dx$

B $-\int_{-1}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx$

C $-\int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx - \int_{2}^{1} f(x)\,dx$

D $\int_{0}^{1} f(x)\,dx - 2\int_{1}^{2} f(x)\,dx$

E $\int_{0}^{1} f(x)\,dx$

2 Find the area under the curve $y = 2x^3$ from $x = 0$ to $x = 1$.

Mastery

3 WORKED EXAMPLE 13 Find the area enclosed between the curve $y = x^2$ and the line $y = x + 6$.

4 Using CAS 5 Find the area enclosed between the graphs of the functions

a $y = x^2 - 5x$ and $y = -x$ b $f(x) = x^2 + 5x$ and $f(x) = x$ c $y = x^2 + 5x$ and $y = -x^2$

5 Find the area enclosed between each pair of graphs.

a $y = 2$ and $y = x^2 + 1$

b $y = x^2$ and $y = -6x + 16$

c $y = 2x^2 - 12x + 20$ and $2x + y = 12$

Calculator-free

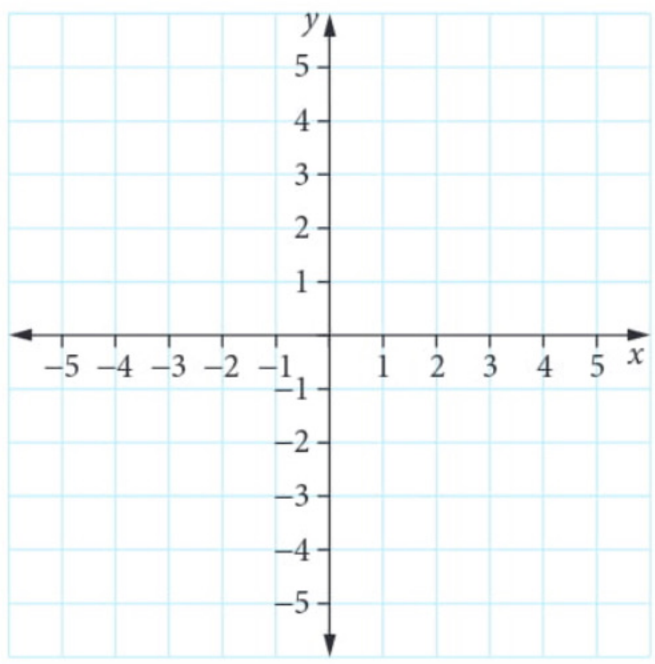
6 (7 marks) Consider the function $f(x) = 3x^2 - x^3$.

a Find the coordinates of the stationary points of the function. (2 marks)

b Copy the axes below and sketch the graph of $f(x)$. (2 marks)

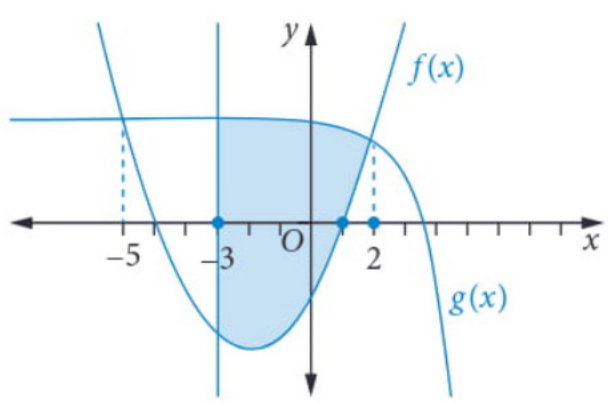
Exam hack

Make sure your graph has rounded turning points and not V-shaped sharp points.

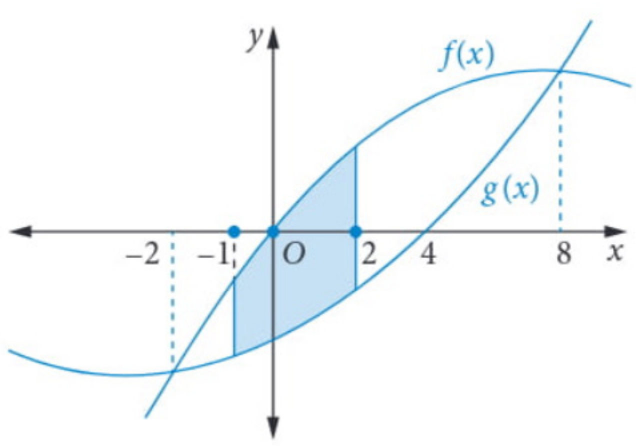


c Find the area enclosed by the graph of the function and the horizontal line given by $y = 4$. (3 marks)

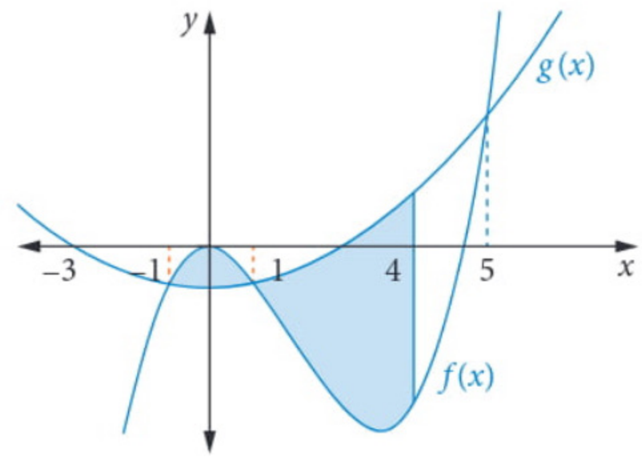
7 (2 marks) Write an integral expression that would determine the shaded area bounded by the curves $y = f(x)$, $y = g(x)$ and the line $x = -3$.



8 (2 marks) Write an integral expression that would determine the shaded area shown.



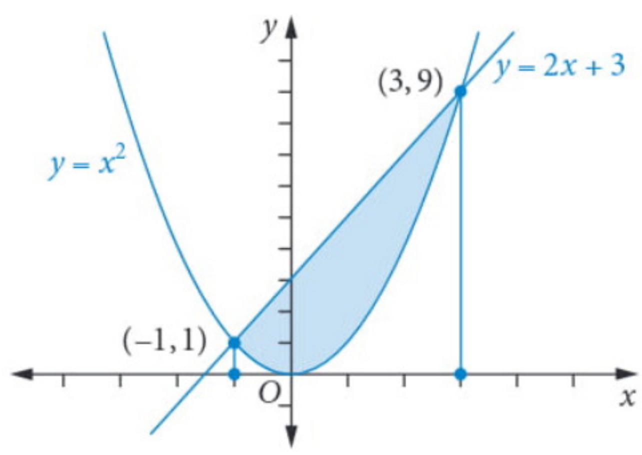
9 (2 marks) Write an integral expression that would determine the total shaded area shown.



10 (3 marks) Determine the area enclosed between the curves $f(x) = (x + 1)(x - 1)$ and $g(x) = x + 1$.

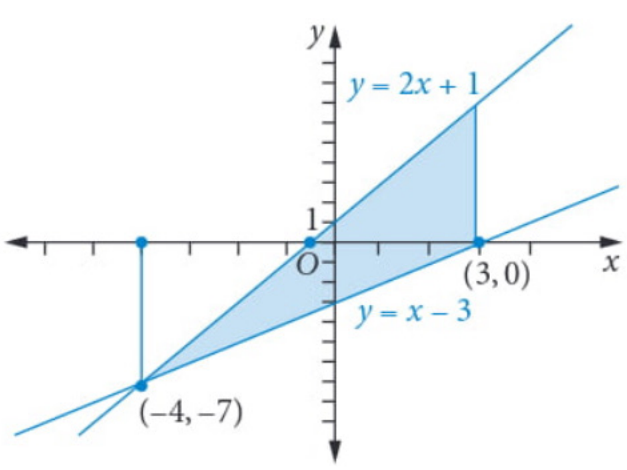
11 (12 marks) Calculate each shaded area.

a



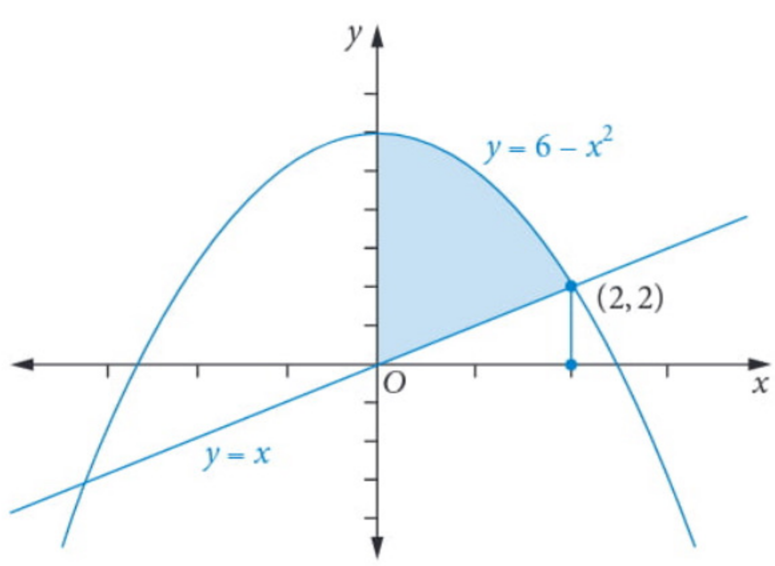
(3 marks)

b



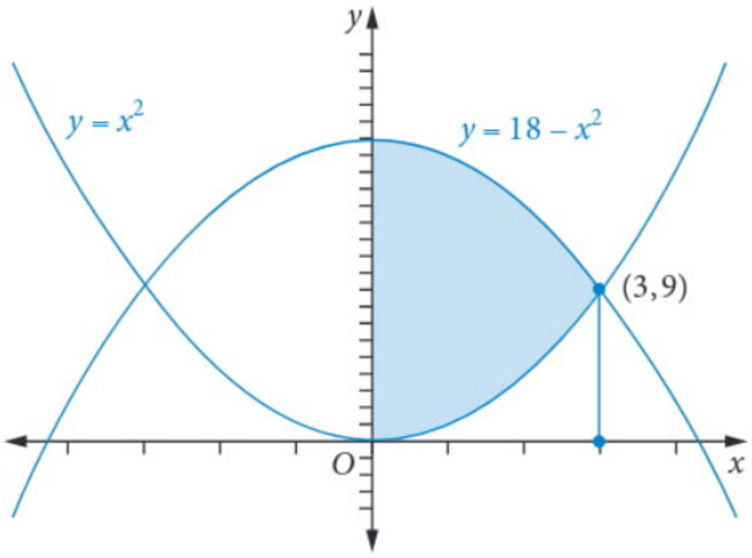
(3 marks)

c



(3 marks)

d



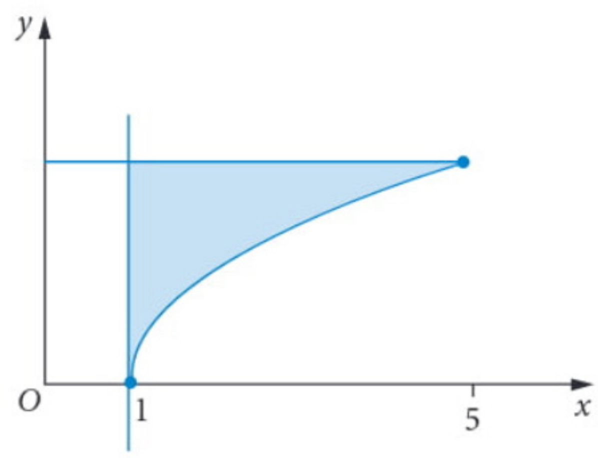
(3 marks)

Calculator-assumed

12 (3 marks) Determine the area enclosed by the two curves $f(x) = x^2 - 3x + 1$ and $y = 2x + 1$.

13 (3 marks) Determine the area between the curves $f(x) = (x + 1)(x^2 - 1)$ and $g(x) = x - 1$.

14 (3 marks) The graph of part of the function of $f(x) = \sqrt{x-1}$ is shown. Determine the area of the shaded region.



15 (4 marks) Parts of the graphs of the functions

$f(x) = x^3 - ax$ $\quad a > 0$

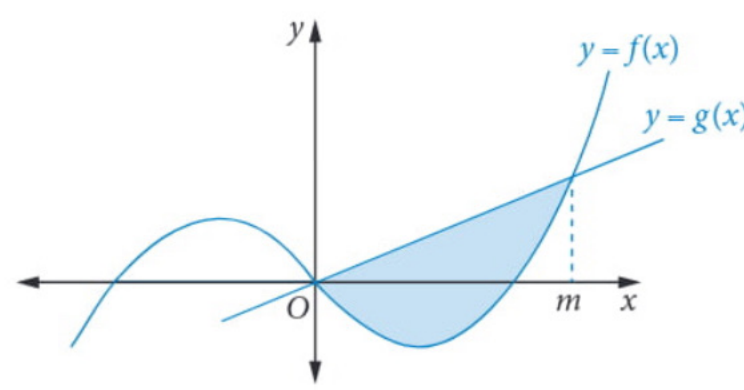
$g(x) = ax$ $\quad a > 0$

are shown in the diagram.

The graphs intersect when $x = 0$ and when $x = m$.

The area of the shaded region is 64.

Find the value of a and the value of m.



3.6 Straight line motion

Video playlist
Straight line motion

Worksheet
Displacement, velocity and acceleration

Straight line motion

Kinematics is the study of motion using **displacement** (x), **velocity** (v) and **acceleration** (a), all in terms of time t, where $t \geq 0$.

We know that $v(t) = \dfrac{dx}{dt}$ and $a(t) = \dfrac{dv}{dt}$.

Therefore, displacement, $x = \int v(t)\,dt$ and velocity, $v = \int a(t)\,dt$.

Displacement is a 'signed distance' that can be positive or negative.

Velocity is a 'signed speed' that can be positive or negative.

Displacement		Velocity		Acceleration
$x, \int v\,dt$	$\rightarrow$ / $\leftarrow$	$v, \dfrac{dx}{dt}, \int a\,dt$	$\rightarrow$ / $\leftarrow$	$a, \dfrac{dv}{dt}, \dfrac{d^2x}{dt^2}$

WORKED EXAMPLE 14 Calculating straight line motion

The acceleration of a particle travelling in a straight line is given by $a(t) = 2t$.

a Find an expression for the velocity $v(t)$ if $v = 0$ when $t = 3$.

b Find an expression for the displacement $x(t)$ if the particle started at the origin.

Steps	Working
a 1 State the rule for acceleration.	$a(t) = 2t$
2 Use velocity $= v = \int a(t)\,dt$.	$v = \int (2t)\,dt$ $\therefore v = t^2 + c$
3 Find c using $v = 0$ when $t = 3$.	$0 = 9 + c$ $\therefore c = -9$
4 State the velocity function.	$v(t) = t^2 - 9$
b 1 Use displacement $= x = \int v(t)\,dt$.	$x = \int (t^2 - 9)\,dt$ $\therefore x = \dfrac{t^3}{3} - 9t + d$ Use d for the constant because we have already used c.
2 Find d using $x = 0$ when $t = 0$.	$x = \dfrac{0^3}{3} - 9(0) + d$ $\therefore d = 0$
3 State the displacement function.	$x(t) = \dfrac{t^3}{3} - 9t$

WORKED EXAMPLE 15 Calculating total distance travelled by an object

A particle moves in a straight line such that its acceleration after t seconds is given by $a(t) = 5t - 6\,\text{m/s}^2$. After 2 seconds. the velocity of the particle is 0 m/s.

a Determine an expression for the velocity in terms of t.

b Determine the change in the displacement of the particle between $t = 0$ and $t = 1$. Interpret your answer.

c Determine the total distance travelled by the particle between $t = 0$ and $t = 1$.

Steps	Working
a 1 Integrate acceleration to find velocity.	$\int (5t - 6)\,dt = \dfrac{5t^2}{2} - 6t + c$
2 Use the given conditions to find the value of c.	$0 = \dfrac{5(2)^2}{2} - 6(2) + c \Rightarrow c = 2$ Therefore, $v(t) = \dfrac{5t^2}{2} - 6t + 2$.
b 1 Determine change in displacement.	$\int_0^1 \left(\dfrac{5t^2}{2} - 6t + 2\right) dt = -\dfrac{1}{6}\,\text{m}$
2 Interpret your answer.	After one second, the particle has moved $\dfrac{1}{6}$ m to the left of the origin.

c 1 Determine whether the particle stops and changes direction. That is, does the velocity equal zero between $t = 0$ and $t = 1$?	$0 = \frac{5t^2}{2} - 6t + 2$ $\therefore t = \frac{2}{5}, 2$
2 Integral needs to be split at $t = \frac{2}{5}$.	$\text{distance} = \int_0^{\frac{2}{5}} \left(\frac{5t^2}{2} - 6t + 2\right) dt - \int_{\frac{2}{5}}^{1} \left(\frac{5t^2}{2} - 6t + 2\right) dt$ $= \frac{137}{150} \approx 0.913\,\text{m}$

As shown in the above example, there are some important points to note.

- Total distance and displacement will not always give you the same answer. It is important to check if the particle has changed direction (that is, velocity is zero) at any stage.
- The integral $\int_{\frac{2}{5}}^{1} \left(\frac{5t^2}{2} + 6t + 2\right) dt$ was subtracted because its area was below the axis if graphed.
- Distance travelled must always be positive, but displacement can be negative.
- It was not necessary to determine an expression for displacement, as we were integrating velocity using definite integrals.
- Part **c** could be done by using one integral if absolute value signs were used.

  For example, $\int_0^1 \left| \frac{5t^2}{2} - 6t + 2 \right| dt = \frac{137}{150}$ m.

WACE QUESTION ANALYSIS

© SCSA MM2016 Q19 Calculator-assumed (8 marks)

The displacement in centimetres of a particle from the point O in a straight line is given by $x(t) = \frac{1}{3}\left(\frac{t}{2} - 4\right)^2 - 2$ for $0 \le t \le 10$, where t is measured in seconds.

Calculate the:

a time(s) that the particle is at rest. (2 marks)

b displacement of the particle during the fifth second. (2 marks)

c maximum speed of the particle and the time when this occurs. (2 marks)

d total distance travelled in the first 10 seconds. (2 marks)

Video WACE question analysis: Integrals

Reading the question

- In part **a**, if the particle is at rest, then the velocity must be equal to zero. The 'times(s)' indicates there may be more than one answer.
- In part **b**, your answer may be negative. Note that it is the 'fifth' second, and not after five seconds.
- In part **c**, care needs to be taken in using the acceleration as this is constant. Therefore, the velocity function (and perhaps a graph) needs to be considered.
- In part **d**, the final answer needs to be positive. Ensure you are finding total distance, and not displacement.

Thinking about the question

- As each question is only worth 2 marks, you do not have to show any working to get full marks. However, it is advised to show some working, so part marks can be attained even if final answer is incorrect.
- As this is a calculator-assumed question, use CAS to calculate integrals and solve equations. The graphing facility may also be useful.

Worked solution (✓ = 1 mark)

a $\frac{dx}{dt} = \frac{1}{3}\left(\frac{t}{2} - 4\right) = 0$

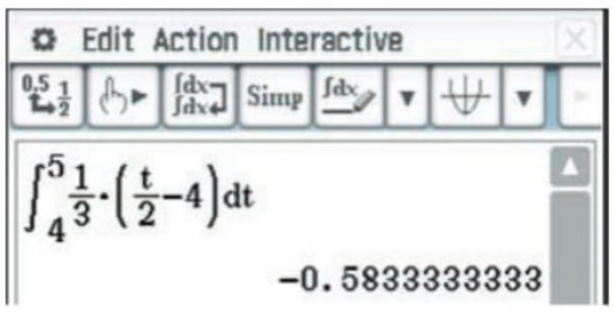
$\frac{t}{2} = 4$

$t = 8$

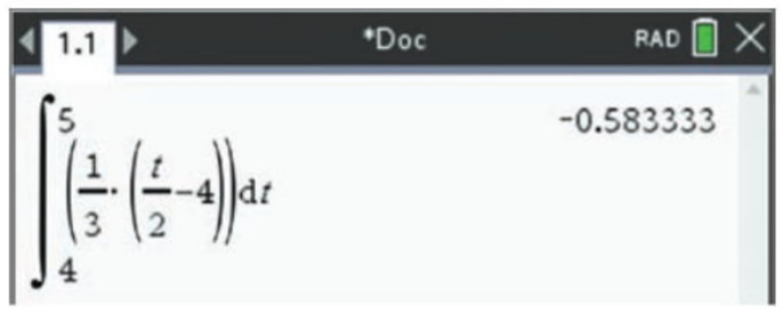
differentiates to determine velocity ✓

solves for time that velocity equals zero ✓

b **ClassPad**



TI-Nspire

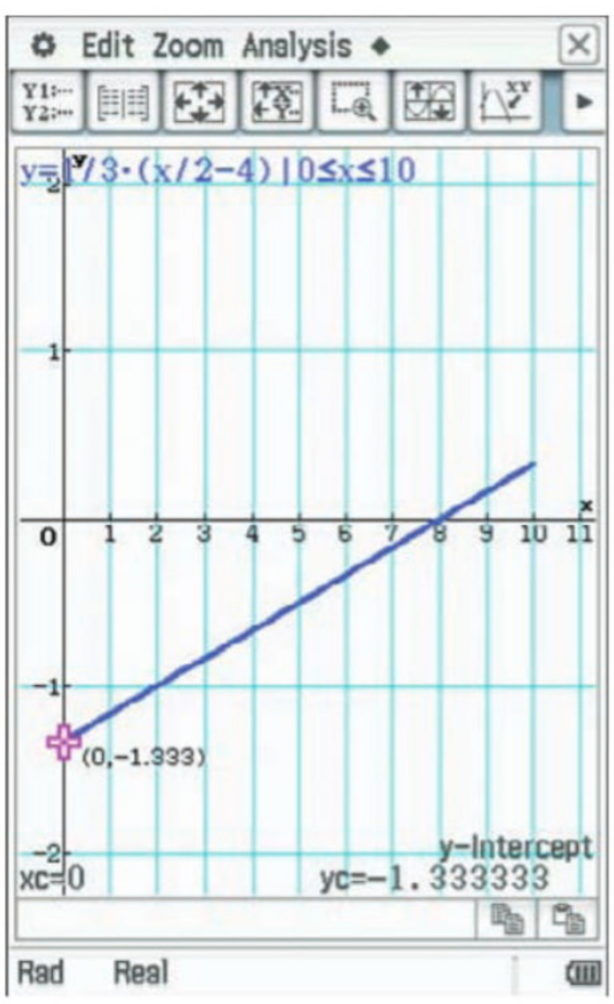


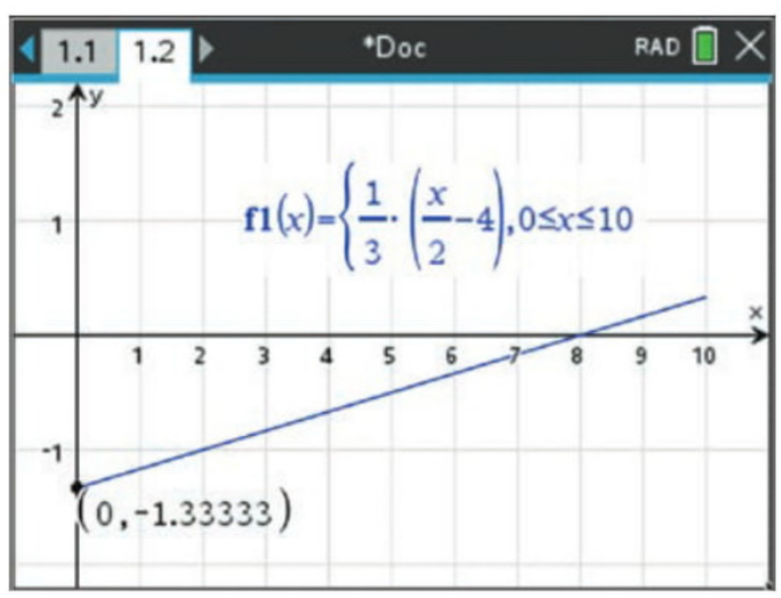
Displacement = −0.5833 cm

examines motion between $t = 4$ and $t = 5$ ✓

determines change in displacement ✓

c $\frac{dx}{dt} = \frac{1}{3}\left(\frac{t}{2} - 4\right), 0 \le t \le 10$



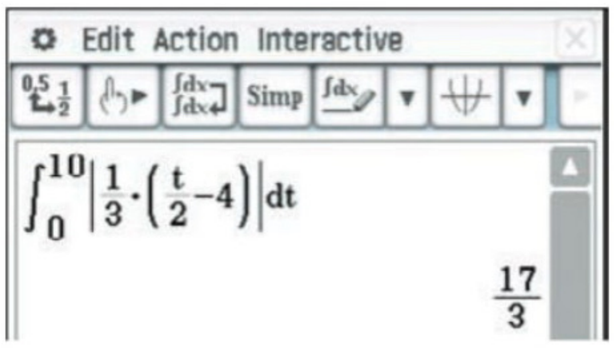


Maximum speed = $\frac{4}{3}$ cm/s at $t = 0$.

examines velocity at endpoints $t = 0$, 10 seconds ✓

determines maximum speed ✓

d ClassPad



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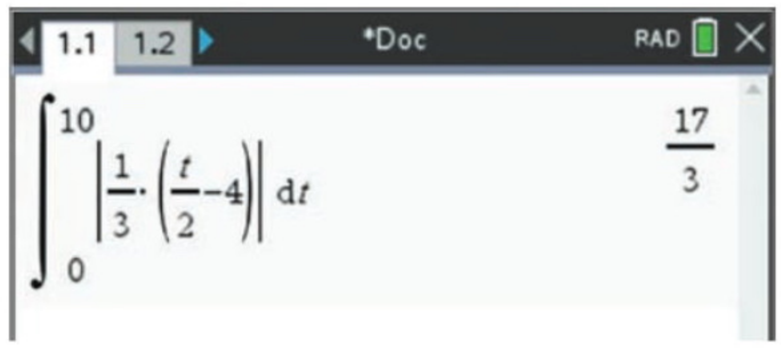


sets up an integral to determine distance travelled ✓

determines distance travelled ✓

or

$t = 10$

$\longrightarrow x = \frac{5}{3}$

$t = 8 \longleftarrow t = 0$

$x = -2 \qquad x = \frac{10}{3}$

Distance travelled $= \frac{15}{3} + \frac{1}{3} + \frac{1}{3} = \frac{17}{3}$

sets up a pathway of motion in first 10 seconds ✓

determines distance travelled ✓

EXERCISE 3.6 Straight line motion

ANSWERS p. 393

Recap

1 The area under the curve $y = x^2 + 1$ from $x = 0$ to $x = 3$, in units^2, equals

A 0 **B** 3 **C** 6 **D** 9 **E** 12

2 The area between the curves $f(x) = x^2 + 1$ and $g(x) = 5$, in units^2, is

A $-\frac{32}{3}$ **B** $\frac{32}{3}$ **C** 5 **D** $\frac{88}{3}$ **E** $\frac{28}{3}$

Mastery

3 WORKED EXAMPLE 14 An object's acceleration as a function of time, t, is given by $a(t) = 4t + 1$. Find its velocity $v(t)$ if the object starts at rest.

4 WORKED EXAMPLE 15 A particle moves in a straight line such that its acceleration after t seconds is given by $a(t) = 3 - 2t$ m/s^2. After 3 seconds, the velocity of the particle is 2 m/s.

a Determine an expression for the velocity in terms of t.

b Determine the change in the displacement of the particle between $t = 1$ and $t = 2$. Interpret your answer.

c Determine the total distance travelled by the particle between $t = 1$ and $t = 2$.

5 The velocity of a particle travelling in a straight line is given by $v(t) = -t^2 + t$. Determine an expression for the displacement $x(t)$ if $x = 0$ when $t = 2$.

6 If the velocity of an object in m/s is described by $v(t) = 3t^2 + 4t^3$ and we know that the object starts at the origin, determine its displacement, in metres.

Calculator-free

7 (2 marks) Determine the velocity of an object which has an acceleration of $a(t) = 4t\,\text{m/s}^2$, given the object stops momentarily at $t = 2$.

8 (2 marks) Determine the displacement of an object which has an acceleration of $a(t) = 3 - 4t\,\text{m/s}^2$, and the object starts at rest from the origin.

9 (7 marks) When a train leaves the station, its velocity is 8 m/s and it accelerates constantly at $0.4\,\text{m/s}^2$.

a Find an expression for the train's velocity as a function of time. (2 marks)

b Find an expression for its displacement. (2 marks)

c Find its displacement after 10 seconds. (3 marks)

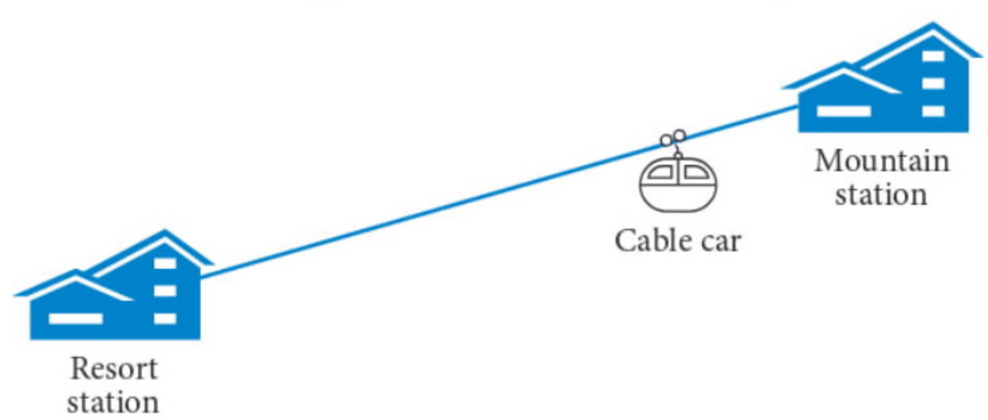
Calculator-assumed

10 © SCSA MM2021 Q14 (5 marks) The displacement in metres, $x(t)$, of a power boat t seconds after it was launched is given by:

$$x(t) = \frac{5t(t^2 - 15t + 48)}{6},\ t \geq 0$$

How far has the power boat travelled before its acceleration is zero?

11 © SCSA MM2021 Q17 (8 marks) A resort in the Swiss Alps features a cable car that travels from the resort station to the mountain station. Engineers are fixing a cable car that unexpectedly stopped shortly before it reached the mountain station. The engineers are ready to test the cable car. For the purposes of the test, the cable car will initially be at rest in its current position, will head up the mountain, stop at the mountain station and immediately return to the resort station where it will stop, and the test will be complete.



The test begins and engineers believe that the acceleration, $a(t)$, of the cable car during the test will be: $a(t) = kt^2 - 23t + 20k$, measured in m/min^2. The variable t is the number of minutes from the moment the cable car leaves its position and k is a constant. After two minutes, the engineers expect that the cable car will be travelling with velocity 18 m/min and will not yet have reached the mountain station.

a Determine the value of the constant k. (3 marks)

b Once the cable car leaves the mountain station, how long should it take to return to the resort station? (3 marks)

c Unfortunately, 10 minutes into the test, the cable car breaks down again. According to the engineers' model, how far is the cable car from the mountain station at this time? (2 marks)

12 (7 marks) A go kart slows down from an initial velocity of 16 m/s until it is stationary. During this interval, its acceleration (t seconds) after the brakes were applied is given by

$$a(t) = \frac{t}{2} - 4\,\text{m/s}^2$$

a Determine the velocity of the vehicle after four seconds. (3 marks)

b Calculate the distance travelled by the vehicle in the time between the brakes being applied and it becoming stationary. (4 marks)

3 Chapter summary

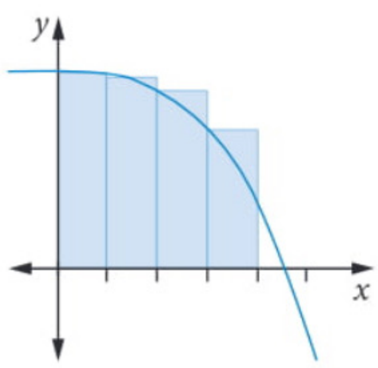
Integrals

The anti-derivative

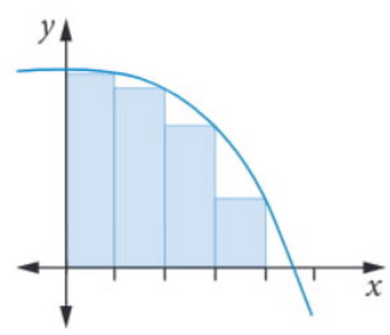
- $F(x)$ is the **anti-derivative** or indefinite **integral** or **primitive** of $f(x)$ if $F'(x) = f(x)$.
- $\int ax^n dx = \dfrac{ax^{n+1}}{n+1} + c$, where ax^n, is called the **integrand**.
- $\int (ax+b)^n dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + c$

Approximating areas under curves

- We can estimate the area under a graph using a series of rectangles.



Using rectangles to overestimate the area



Using rectangles to underestimate the area

The definite integral

- A definite integral can be evaluated using $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F(x)$ is an anti-derivative of $f(x)$.

Properties of the definite integral

1 $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	**Constant out:** a constant factor can be taken out of an integral.
2 $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	**Split terms:** a sum or difference of terms can be integrated separately.
3 If b is between a and c, then: $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$	**Split limits:** the limits of an integral can be split.
4 $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$	**Swap limits:** reversing the order of the limits changes the sign of the definite integral.
5 $\int_a^a f(x)\,dx = 0$	**Same limits:** gives $F(a) - F(a) = 0$.

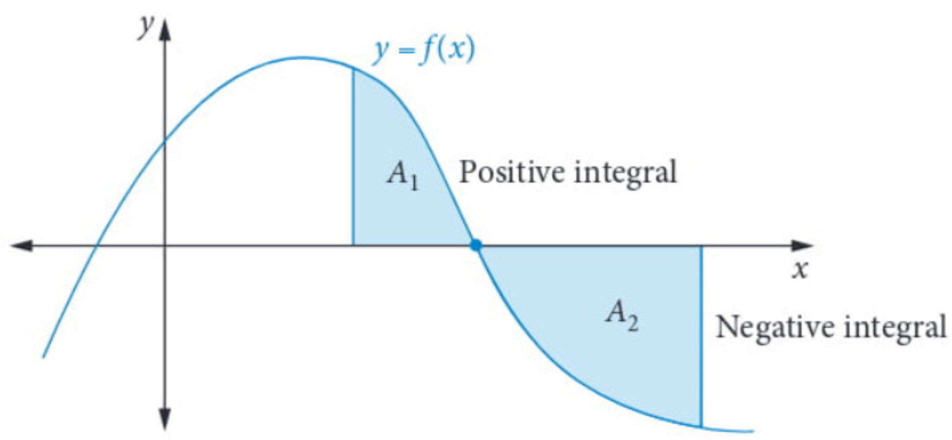
The fundamental theorem of calculus

The two parts of the fundamental theorem are

$$F'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x) \quad \text{and} \quad \int_a^b f'(x)dx = f(b) - f(a)$$

Area under a curve

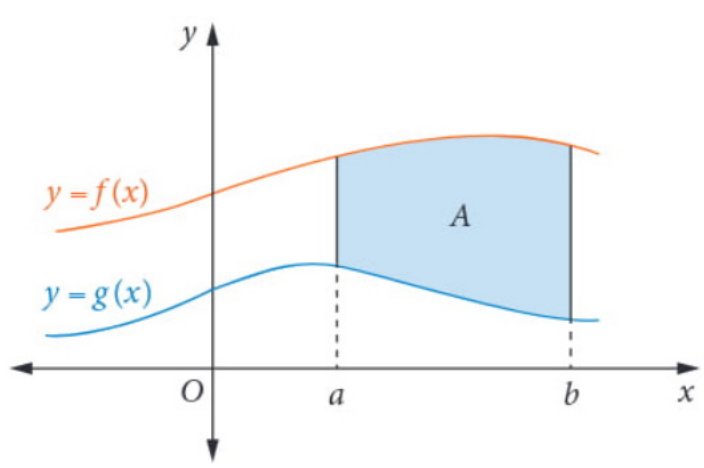
- The definite integral $\int_a^b f(x)\,dx$ gives the **signed area** enclosed by the graph of $y = f(x)$ and the x-axis between $x = a$ and $x = b$.
- If the section of the graph is above the x-axis, then the definite integral gives the area under the curve.
- If the section of the graph is below the x-axis, then the definite integral gives the area above the curve and its value is negative.



- If we need to find areas using integration, we need to check whether the graph goes below the x-axis and change the sign for areas below the x-axis (to make them positive).

Areas between curves

- The area can be calculated as the difference between the areas under the two functions regardless of the position of the area.



- area = $\int_a^b (\text{upper} - \text{lower})\,dx = \int_a^b [f(x) - g(x)]\,dx$

Straight line motion

- velocity = $v = \int a(t)\,dt$
- displacement = $x = \int v(t)\,dt$

Cumulative examination: Calculator-free

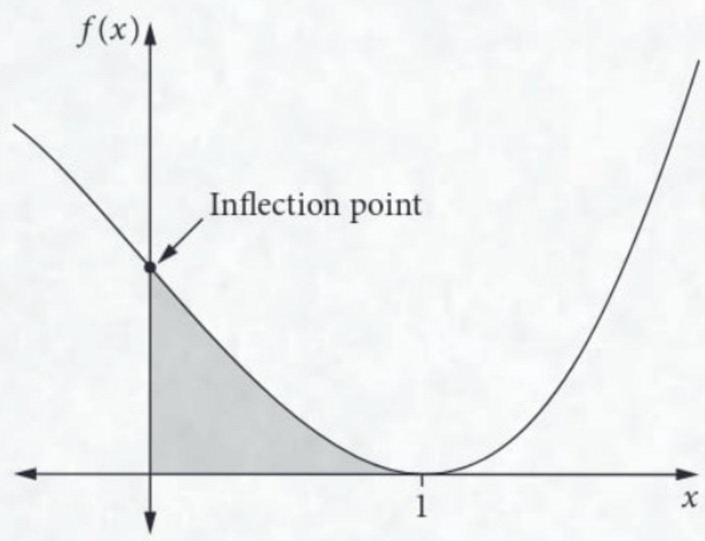
Total number of marks: 26 Reading time: 3 minutes Working time: 26 minutes

1 (2 marks) If $f(x) = \int_0^x (\sqrt{t^2 + 4})\,dt$, determine the value of $f'(-2)$.

2 (2 marks) Evaluate $\int_1^2 \left(3x^2 - \frac{1}{x^2}\right) dx$.

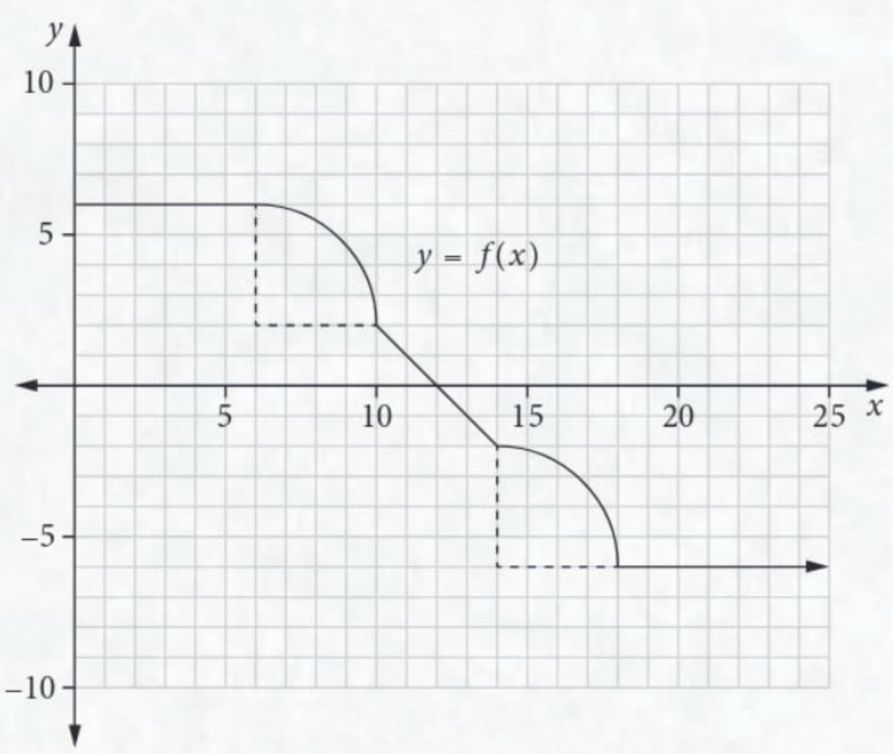
3 (2 marks) Find $f(x)$ given that $f(4) = \frac{64}{3}$ and $f'(x) = x^2 - 10x - x^{-\frac{1}{2}} + 1, x > 0$.

4 © SCSA MM2020 Q3 (7 marks) The graph of the cubic function $f(x) = ax^3 + bx^2 + cx + d$ is shown. A turning point is located at $(1, 0)$ and the shaded region shown on the graph has an area of $\frac{3}{2}$ units^2.



Use the above information to determine the values of a, b, c and d.

5 © SCSA MM2016 Q7 (7 marks) Consider the graph $y = f(x)$. Both arcs have a radius of four units.



Using the graph of $y = f(x)$, $x \geq 0$, evaluate exactly the following integrals.

a $\int_0^{12} f(x)\,dx$ (3 marks)

b $\int_0^{18} f(x)\,dx$ (2 marks)

c Determine the value of the constant α such that $\int_0^{\alpha} f(x)\,dx = 0$. There is no need to simplify your answer. (2 marks)

6 © SCSA MM2021 Q5 (6 marks)

a Determine the area between the parabola $y = x^2 - x + 3$ and the straight line $y = x + 3$. (4 marks)

b The area between the parabola $y = x^2 - x - 2$ and the straight line $y = x - 2$ is the same as the area determined in part **a**. Explain why this is the case. (2 marks)

Cumulative examination: Calculator-assumed

Total number of marks: 27 Reading time: 3 minutes Working time: 27 minutes

1 (2 marks) Consider $f(x) = x^2 + \frac{p}{x}, x \neq 0$. There is a stationary point on the graph of f when $x = -2$. Determine the value of p.

2 (3 marks) Determine the area enclosed between the graph of $y = -2\sqrt{x-1}$, the x-axis and the line $x = 2$.

3 © SCSA MM2017 Q20 (9 marks) A model train travels on a straight track such that its acceleration after t seconds is given by $a(t) = pt - 13\,\text{cm/s}^2$, $0 \leq t \leq 10$, where p is a constant.

a Determine the initial acceleration of the model train. (1 mark)

The model train has an initial velocity of 5 cm/s. After 2 seconds it has a displacement of −50 cm. A further 4 seconds later its displacement is 178 cm.

b Determine the value of the constant p. (4 marks)

c When is the model train at rest? (2 marks)

d How far has the model train travelled when its acceleration is $47\,\text{cm/s}^2$? (2 marks)

4 © SCSA MM2018 Q16 (8 marks) Let $f(x)$ be a function such that $f(-2) = 4, f(-1) = 0, f(0) = -1, f(1) = 0$ and $f(3) = 2$. Further, $f'(x) < 0$ for $-2 \leq x < 0$, $f'(0) = 0$ and $f'(x) > 0$ for $0 < x \leq 3$.

a Evaluate the following definite integrals:

i $\int_0^3 f'(x)\,dx$. (2 marks)

ii $\int_{-2}^3 f'(x)\,dx$. (2 marks)

b What is the area bounded by the graph of $f'(x)$ and the x-axis between $x = -2$ and $x = 3$? Justify your answer. (4 marks)

5 (5 marks) Consider functions $f(x) = \frac{81x^2(a-x)}{4a^4}$ and $h(x) = \frac{9x}{2a^2}$, where a is a positive real number.

a Find the coordinates of the local maximum of $f(x)$ in terms of a. (2 marks)

b Find the x values of all the points of intersection between the graphs of $f(x)$ and $h(x)$, in terms of a where appropriate. (1 mark)

c Determine the total area of the regions bounded by the graphs of $y = f(x)$ and $y = h(x)$. (2 marks)